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Completeness of Boolean powers of Boolean algebras

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Introduction.

In [4], Dwinger considered the completeness of Boolean powers of complete Boolean algebras. Dwinger obtained a necessary and sufficient condition in algebraic form:

THEOREM (Dwinger [4]). *Let A and B be complete Boolean algebras. The Boolean power $A[B]$ is complete if and only if*

$$\bigvee_{x\in A}\Big(\bigwedge_{y\nleq x}\sim f(y)\wedge\bigwedge_{x\nleq z}\ \bigvee_{u\nleq z}f(u)\Big)=\mathbf{1}\qquad\text{for each}\quad f:A\longrightarrow B.$$

Our main purpose is to consider some relationship which exists among the completeness of $A[B]$, the distributive-like properties of B and the saturation number of A.

In the notation of Boolean valued models of set theory, the Boolean power $A[B]$ is isomorphic to

$$\hat{\check{A}}=\{f\in V^{(B)}\mid [\![f\in\check{A}]\!]^{(B)}=\mathbf{1}\}$$

where $\check{A}$ is an element of $V^{(B)}$ such that $\check{A}=\{\check{a}\mid a\in A\}\times\{\mathbf{1}\}$. We can have a better perspective, if we deal with $\check{A}$ in $V^{(B)}$ instead of $A[B]$. By virtue of 5.5 of Solovay and Tennenbaum [13], $A[B]$ is complete if and only if $[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$. Since

$$\begin{aligned}&[\![\check{A}\text{ is complete}]\!]^{(B)}\\&=[\![\forall X\subset\check{A}\ \exists x\in\check{A}[\forall y\in X[y\leqq x]\wedge\forall z\in\check{A}\ [\forall u\in X[u\leqq z]\Rightarrow x\leqq z]]]\!]^{(B)}\\&=\bigwedge_{f:A\to B}\Big(\bigvee_{x\in A}\Big(\bigwedge_{y\nleq x}\sim f(y)\wedge\bigwedge_{x\nleq z}\ \bigvee_{u\nleq z}f(u)\Big)\Big),\end{aligned}$$

we can obtain a proof of Dwinger's theorem which uses Boolean valued models of set theory. This suggests why we are going to work in $V^{(B)}$. We assume that the reader is familiar with the technique of Boolean valued models of set theory, as presented, e.g. [6, 7, 13]. We assume that $V^{(B)}$ is separated, i.e.,