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Fractional powers of operators

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1. Introduction.

The study of fractional powers of operators has an extensive history. The semigroup of fractional powers of a bounded operator was studied by Hille [5] (1939). Fractional powers for the negatives of the infinitesimal generators of bounded strongly continuous semigroups have been discussed by Bochner [3], Phillips [19], Yosida [22] and Balakrishnan [1], who afterwards (1960) [2] gives a new definition and extends this theory to closed linear operators A in a Banach space X such that $]-\infty, 0[$ is contained in the resolvent set $\rho(A)$, and the resolvent satisfies

$$\|\mu(\mu+A)^{-1}\| \leqq M < \infty, \qquad 0<\mu<+\infty \tag{1.1}$$

(operators which we shall call non-negatives, following the terminology used by H. Komatsu [13]).

Balakrishnan [2] defines the power with base A and exponent α ($\mathrm{Re}\,\alpha>0$) as the closure of a closable operator, J^α, whose expression is:

$$\left\{\begin{array}{l}
\text{For } 0<\mathrm{Re}\,\alpha<1,\ D(J^\alpha)=D(A), \\
\qquad J^\alpha\varphi = \dfrac{\sin\alpha\pi}{\pi}\displaystyle\int_0^{+\infty}\mu^{\alpha-1}(\mu+A)^{-1}A\varphi d\mu . \\
\text{For } 0<\mathrm{Re}\,\alpha<2,\ D(J^\alpha)=D(A^2), \\
\qquad J^\alpha\varphi = \dfrac{\sin\alpha\pi}{\pi}\displaystyle\int_0^{+\infty}\mu^{\alpha-1}\Big[(\mu+A)^{-1}-\frac{\mu}{1+\mu^2}\Big]A\varphi d\mu+\sin\Big(\alpha\frac{\pi}{2}\Big)A\varphi . \\
\text{For } n<\mathrm{Re}\,\alpha<n+1,\ \ D(J^\alpha)=D(A^{n+1}), \\
\qquad J^\alpha\varphi = J^{\alpha-n}A^n\varphi . \\
\text{For } n<\mathrm{Re}\,\alpha\leqq n+1,\ \ D(J^\alpha)=D(A^{n+2}), \\
\qquad J^\alpha\varphi = J^{\alpha-n}A^n\varphi .
\end{array}\right. \tag{1.2}$$

This author proves that

$$\overline{\overline{J^\alpha}\,\overline{J^\beta}} = \overline{J^{\alpha+\beta}} .$$