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An asymptotic estimation of dimension of harmonic spinors

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§ 1. Introduction.

The purpose of the present paper is to give an asymptotic bounds of the dimension of twisted harmonic spinors.

Let (X, g) be an oriented $2n$-dimensional compact spinnable Riemannian manifold, and $\{L, h\}$ a C^∞ complex line bundle over X with a Hermitian fibre metric h. We consider the twisted Dirac's operator $D_k: \Gamma(S\otimes L^k)\to\Gamma(S\otimes L^k)$ which is naturally induced from the Levi-Civita connection of (X, g) and the Hermitian connection of $\{L, h\}$. Here S denotes the spinor bundle of X. Let Δ_k be the Laplace-Beltrami operator of D_k. Then, by Bochner-Weizenböck formula, we obtain that, for $u\in\Gamma(S\otimes L^k)$,

$$\int_X\langle u, \Delta_k u\rangle dV_g=\int_X\left\{|\nabla_k u|^2+\frac{\kappa}{4}|u|^2+k\langle\hat{\Theta}_h u, u\rangle\right\}dV_g,$$

where $\langle\,,\rangle$ represents the inner product on $S\otimes L^k$ with respect to the metric g and h, κ is the scalar curvature of (X, g), and $\hat{\Theta}_h$ is an element of $\mathrm{End}_C(S\otimes L^k)$ which is defined as

$$\hat{\Theta}_h:=\frac{1}{2}\sum_{i,j}(e_ie_j\otimes\Theta_h(e_i, e_j)).$$

Here $\{e_1, \cdots, e_{2n}\}$ is an oriented orthonormal base of T_xX, and Θ_h is the curvature form of $\{L, h\}$. Now, following Demailly's observation ([3]), we consider the operator $\kappa/4+k\hat{\Theta}_h$ as a potential of the Dirac's operator D_k, and we shall show that the dimension of harmonic spinors of D_k can be asymptotically estimated in terms of the operator $\hat{\Theta}_h$ as k goes to infinity. In fact, using Theorem 2.3 of [3], we shall show the following asymptotic estimation which is a Dirac's operator-version of Demailly's result on $\bar{\partial}$-operator.

THEOREM. *For the curvature form Θ_h of $\{L, h\}$, we define a subset X_+ (resp. X_-) of X as*

$$X_+\ (resp.\ X_-):=\{x\in X|\ ((i\Theta_h)^n/dV_g)(x)>0\ (resp.\ <0)\},$$

where dV_g is the volume form of (X, g), and we define $H_k^+(0)$ (resp. $H_k^-(0)$) as