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Isotropy representations of semisimple symmetric spaces and homogeneous hypersurfaces

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The isotropy representation of a symmetric space is the linear action of the isotropy group of a point on the tangent space. This paper deals with the case that the orbits have codimension 2, i.e. they are homogeneous hypersurfaces in the pseudo-riemannian sphere. In the Riemannian case all homogeneous hypersurfaces of the sphere are orbits under isotropy representations and Takagi/Takahashi [**13**] have studied their geometry in detail. These investigations will be extended to the isotropy representations of semisimple symmetric spaces.

The first section gives the algebraic prerequisites about semisimple symmetric Lie algebras and their isotropy representations.

Section 2 contains the geometric results. Hypersurfaces in the pseudo-riemannian sphere occur as orbits under the isotropy representation if the symmetric Lie algebra is of rank 2. The hypersurfaces have 2, 3, 4 or 6 distinct principal curvatures. Depending on the rank of the maximal compact symmetric subalgebra, either the orbits form one family of homogeneous hypersurfaces with complex principal curvatures and at most one focal variety, or they form several families, one with real principal curvatures and at least 2 focal varieties and the other families with only one focal variety that coincides with a focal variety of the first family.

The last section gives a list of the examples and their geometric data: principal curvatures and the numbers of focal varieties and of families of homogeneous hypersurfaces. Finally an example of a homogeneous hypersurface is displayed that is *not* orbit under an isotropy representation.

The material is taken from the author's Bonn University doctoral dissertation [**5**], which may be consulted for detailed proofs.

1. Semisimple symmetric Lie algebras.

A *semisimple symmetric Lie algebra* $(\underline{g}, \sigma)$ consists of a real semisimple Lie algebra $\underline{g}$ and an involution σ. Let $\underline{g}=\underline{h}+\underline{q}$ be the decomposition into eigenspaces of σ, i.e. $\underline{h}=\{X\in\underline{g} \mid \sigma X=X\}$ and $\underline{q}=\{X\in\underline{g} \mid \sigma X=-X\}$. Then $[\underline{h}, \underline{h}]\subset\underline{h}$, i.e. $\underline{h}$ is a subalgebra of $\underline{g}$, $[\underline{h}, \underline{q}]\subset\underline{q}$ and $[\underline{q}, \underline{q}]\subset\underline{h}$. The notation $(\underline{g}, \sigma)=\underline{g}/\underline{h}$