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Ruled fibrations on normal surfaces

Dedicated to Professor M. Nagata on his 60th birthday

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Let Y be a normal projective surface over $\boldsymbol{C}$. A *ruled fibration* on Y over a smooth curve B is a surjective morphism $p: Y \to B$ such that the general fibre is isomorphic to $\boldsymbol{P}^1$. We have the notion of exceptional curves of the first kind in the category of normal surfaces. Namely, an irreducible curve C on Y is called an *exceptional curve of the first kind* if $K_Y C < 0$ and $C^2 < 0$, where the K_Y denotes a canonical divisor on Y. Cf. [**S3**]. A *minimal ruled fibration* will mean a ruled fibration whose fibres contain no exceptional curves of the first kind. Given a ruled fibration on Y, contract successively all exceptional curves of the first kind in fibres, then we obtain a minimal ruled fibration. In this paper we study the structure of a normal surface Y having a minimal ruled fibration over a curve B of genus g.

In § 1 we consider the structure of singular fibres. It turns out that every singular fibre is necessarily a multiple fibre and contains one or two singular points of Y. To describe a singular fibre, we observe the weighted dual graph of the inverse image of the singular fibre on the minimal resolution of Y. In § 2 we introduce a nonnegative rational number τ, which measures the amount of Sing(Y). We have the formula: $K_Y^2 = 8(1-g) - 4\tau$. Suppose that Y has singular fibres f_i with multiplicities m_i, $i = 1, \cdots, k$. Then we show that $\tau \geqq \sum(1 - 1/m_i)$. In § 3 we define the invariants $s_n \in \boldsymbol{Q}$ for positive integers n. The first invariant $s = s_1$ is defined to be the minimum of the self-intersection numbers of all sections in the ruled fibration. Provided that Y is singular, we prove the inequality: $s \leqq g + \tau - 1$. Recall that for the smooth case a theorem of Nagata [**N**] says that $s \leqq g$. Similarly, we define the invariants s_n to be $1/n^2$ of the minimum of the self intersection numbers of all effective divisors of degree n over B. We show that $s_n \leqq 2g/(n+1) + \tau$. The invariant $s_* = \inf\{s_n\}$ plays an important role in the numerical criterion for an ample divisor. In § 4 we consider the anti-Kodaira dimension $\kappa^{-1}(Y)$. We give a classification of Y in terms of $\kappa^{-1}(Y)$ together with the numerical type of the anticanonical divisor $-K_Y$. For the smooth case, this was done in [**S1**], [**S3**]. We also deal with the question when Y admits another ruled fibration or an elliptic fibration. We