J. Math. Soc. Japan
Vol. 40, No. 2, 1988

On the structure of locally convex filtrations on complete manifolds

By Takao YAMAGUCHI

(Received Oct. 20, 1986)

Introduction.

Let M be a connected complete Riemannian manifold without boundary. It is known that the existence of a convex function imposes a strong restriction on the topology of M. In fact, according to Greene and Shiohama [7], [8] and to Bangert [1] for a special case, if M admits a locally nonconstant convex function, then it is diffeomorphic to the normal bundle of a submanifold. In particular, M is noncompact. The author has shown in the previous work [22] that the same conclusion is still valid for a locally quasiconvex function, which is a generalization of convex functions. In this paper, we consider more general function concerning convexity. We say that a continuous function $f: M \to \boldsymbol{R}$ is a *locally convex filtration* if it is locally nonconstant and if all sublevel sets $M^a = \{x ; f(x) \leqq a\}$ are locally convex in M. This is a natural generalization of convex functions and locally quasiconvex functions. It should be noted that some compact manifolds admit such filtrations. For example, the function f on the unit sphere S^n in $\boldsymbol{R}^{n+1}$ defined by $f(x^1, \cdots, x^{n+1}) = -(x^{n+1})^2$ provides such an example. The purpose of the present paper is to characterize the geometric structure of the filtrations, and to classify the topological structure of manifolds admitting the filtrations of a certain type.

Let H_f^* be the union of level components of f intersecting the closure of the local maximum set H_f of f. Under a certain regularity condition on f, we shall prove that *H_f is (if it is not empty) a locally finite union of totally geodesic hypersurfaces and the complement $H_f^* - H_f$ is a Lipschitz submanifold* (Theorem 2.3), and that *each connected component of $M - H_f^*$ is homeomorphic (diffeomorphic if the boundary of the component is smooth) to the normal bundle of a submanifold* (Theorem 3.1). This is an extension of the works [1], [7], [8] and [22] stated in the beginning.

On the other hand, to treat the classification problem, it will be needed to restrict our filtrations to have a nice property because any complete surface has

This research was partially supported by Grant-in-Aid for Scientific Research (No. 61740050), Ministry of Education, Science and Culture.