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Divergent formal solutions to Fuchsian partial differential equations

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§1. Introduction.

Let $x=(x_1, \cdots, x_d)\in \boldsymbol{R}^d$ be the variable in $\boldsymbol{R}^d$ and let us put $\partial=(\partial_1, \cdots, \partial_d)$ where $\partial_j=\partial/\partial x_j$, $j=1, \cdots, d$. For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_d)\in \boldsymbol{N}^d$, $\boldsymbol{N}=\{0, 1, 2, \cdots\}$ we set $(x\cdot\partial)^\alpha=(x_1\partial_1)^{\alpha_1}\cdots(x_d\partial_d)^{\alpha_d}$. Let $\lambda\in \boldsymbol{C}^d$ be given and fixed. Then we shall study the characterization of divergent formal solutions $u(x)$ of the form $u(x)=x^\lambda \sum_{\eta\in \boldsymbol{N}^d} v_\eta x^\eta/\eta!$ of the equation

$$(1.1)\qquad P(x\,;x\cdot\partial)u \equiv \Big(\sum_{|\alpha|=m} a_\alpha(x\cdot\partial)^\alpha+\sum_{|\beta|\leqq m-\sigma} b_\beta(x)(x\cdot\partial)^\beta\Big)u(x) = f(x)x^\lambda$$

where $\sigma\geqq 1$ is an integer, $m\in \boldsymbol{N}$ and a_α's are complex constants. We assume that the function $b_\beta(x)$ is analytic at the origin and that $f(x)$ is a given analytic function.

For ordinary differential equations of Fuchs type (i.e. $d=1$ in (1.1)) we know that all formal solutions of Equation (1.1) converge. Nevertheless, in the case $d\geqq 2$ we often get divergent formal solutions of Equation (1.1) if the coefficients satisfy certain conditions (cf. [**3**], [**9**]). In fact there exist equations with infinite-dimensional kernel and those with small denominators. Typical examples are the equations $(x_1\partial_1-\tau x_2\partial_2)u=f(x)$ where τ is a positive rational and irrational number respectively. By using elementary facts of diophantine analysis we can show that there exists an irrational $\tau>0$ and an entire function $f(x)$ such that the equation for this τ and $f(x)$ has a formal solution $u(x)=\sum u_\eta x^\eta$ with the estimate $|\eta|!^s/|u_\eta|\to 0$ as $|\eta|\to\infty$ for $s=1, 2, \cdots$. In this case the formal solution has bad behavior. Even for these simple examples the criterion which distinguishes such bad equations from good ones can only be expressed by the number-theoretical properties of τ, and is not simple. Hence if we are to study formal solutions of more general equations in the analytic category we need very delicate and complicated arguments (cf. [8], [**9**]). It is an interesting problem to give a meaning to such divergent solutions and to study whether this phenomenon is peculiar to analytic solutions or also occurs for C^∞-solutions.