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Geometry of anti-self-dual connections and Kuranishi map

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1. Introduction.

Let (M, h) be a compact, connected oriented C^{∞} Riemannian 4-manifold. Let P be a C^{∞} $SU(n)$-principal bundle over M, $n \geqq 2$, and $\mathcal{A}^{-}$ the set of anti-self-dual Yang-Mills connections on P. The quotient space $\mathcal{M}^{-}$ of $\mathcal{A}^{-}$ by the action of gauge transformations is called the moduli space of anti-self-dual connections on P.

It is a known fact that the moduli space is a finite dimensional C^{∞} manifold (with singularities). This fact is originally due to [**3**].

On the other hand we have an example of the moduli space over a particular 4-manifold with a complete description. Over the 4-sphere S^4 with the standard metric the moduli space of anti-instantons of index $k=1$ and of group $SU(2)$, namely the set of equivalence classes of BPST-anti-instantons is given by the upper half space $\boldsymbol{R}_{+}^{5}=\{(\xi, \lambda), \xi \in \boldsymbol{R}^{4}, \lambda>0\}$ where ξ and λ are the center and the scale of an anti-instanton ([**2**]). Here the index k denotes $c_2(E)$, $E=P \times_{\rho} \boldsymbol{C}^{2}$.

Moreover the moduli space $\mathcal{M}^{-}$ of $SU(2)$-anti-self-dual connections of index 2 over the complex projective plane $P_2(\boldsymbol{C})$ with the Fubini-Study metric is identified, by theorems of Kobayashi and Lübke ([**16**], [**21**]) and also a theorem of Donaldson ([**9**]), with the moduli of stable holomorphic structures on the associated vector bundle E, which turns out to be the complement of a singular hypersurface of degree 3 in $P_5(\boldsymbol{C})$; $\{(z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \in P_5(\boldsymbol{C})\,;\, z_0 z_1 z_2 - z_0 z_5^2 - z_2 z_3^2 - z_1 z_4^2 + 2 z_3 z_4 z_5 = 0\}$ by a result of [**5**] (see also [**24**], p. 352).

The moduli space over a 4-manifold M is considered to reflect topological properties of the base space M ([**8**], [**10**], [**11**]).

Moreover, since the definition of anti-self-dual connection requires the presence of a Riemannian structure on M, the space $\mathcal{M}^{-}$ must admit a Riemannian structure, for instance, a metrical structure, curvature properties, etc.

When the base space is a complex surface with a Kähler metric, the anti-self-duality of a connection gives an integrability condition of a holomorphic structure on the associated vector bundle and the moduli space enjoys a structure of complex manifold (with singularities) of complex dimension