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Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited

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1. Introduction.

The theory of viscosity solutions of first order scalar equations of the form

$$F(x, u, Du)=0 \quad \text{in } \Omega, \tag{E}$$

where Ω is an open set in $\boldsymbol{R}^n$, $u:\Omega\to\boldsymbol{R}$ is continuous (i.e., $u\in C(\Omega)$), $F\in C(\Omega\times\boldsymbol{R}\times\boldsymbol{R}^n)$ enjoys some monotonicity in u and Du denotes the gradient of u, has undergone a rapid and intensive development. The original uniqueness proofs of M.G. Crandall and P.L. Lions [**4**], [**5**] were recast in M.G. Crandall, L.C. Evans and P.L. Lions [**3**] and then further improvements were made by H. Ishii [**11**], [**12**], [**13**] and others. These and other improvements, however, did not alter the basic structure of the original uniqueness argument in the sense that each significant step in the original argument had its parallel in the modified ones. As we think of it today, if u and v are solutions of (E) in the viscosity sense, then the idea was to consider maximum points of $\Phi(x, y)=u(x)-v(y)-|x-y|^2/\varepsilon$ (or an analogue) over $\bar{\Omega}\times\bar{\Omega}$ and then to reduce to the case that the maximum is an interior point and to use the equation satisfied in the viscosity sense to estimate $u-v$ at such a maximum. Of course, if Ω is unbounded the possibility of the maximum not being attained must be disposed of by adding appropriate terms to Φ, and in all cases suitable use must be made of structure conditions on the equation and so on. In all but exceptionally simple cases, careful estimates on terms corresponding to $(x-y)/\varepsilon$ at the maximum point were made and used in the course of argument.

In this note we change the point of view a bit and emphasize a way of thinking that has evolved in recent papers. The result is simpler proofs of greater generality. This carries over, as well, into other aspects of the theory

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