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Extensions of nonlinear completely positive maps

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Introduction.

It has been well recognized that the most appropriate notion of positivity for linear maps between C^*-algebras is the complete positivity. Although there were classical works [8, **11**, **12**] on numerical completely positive functions, it was not until the recent papers of Ando and Choi [**1**] and Arveson [**3**] that the nonlinear complete positivity was investigated in the C^*-algebraic framework. According to [**1**], in spite of the extent of nonlinearity, any completely positive map between arbitrary C^*-algebras admits a nice representation as a doubly infinite sum of compressions of completely positive linear maps on certain C^*-tensor products. On the other hand, the essentially similar representation was obtained in [**3**] for bounded completely positive complex-valued functions on the open unit ball of a unital C^*-algebra.

Since Arveson's Hahn-Banach type extension theorem [2] for completely positive linear maps, the linear completely positive extension has been discussed especially in connection with injectivity and nuclearity of C^*-algebras (see e.g. [**5**, **7**]). It seems natural to consider the nonlinear counterpart of complete positive extension. The purpose of this paper is to investigate the problem when completely positive maps defined on $\mathcal{A}$ (resp. ball $\mathcal{A}$, the open unit ball of $\mathcal{A}$) can be extended on $\mathcal{B}$ (resp. ball $\mathcal{B}$) given a C^*-subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$.

In Section 1 of this paper, on the lines of [**1**] we generalize the representation theorem in [**3**] to bounded completely positive maps on ball $\mathcal{A}$ with values in a von Neumann algebra. In Section 2, we show the local uniform continuity of completely positive maps. In Section 3, we give some completely positive extension theorems in special cases when $\mathcal{B}=\mathcal{A}_{\mathrm{I}}$ or $\mathcal{A}$ is seminuclear. We further characterize pairs $\mathcal{A}\subset\mathcal{B}$ of C^*-algebras having the completely positive extension property. It is proved above all that every completely positive map from $\mathcal{A}$ to $B(\mathcal{H})$ is extended on $\mathcal{B}$ if and only if $\mathcal{A}^{\otimes m}\otimes\bar{\mathcal{A}}^{\otimes n}\subset\mathcal{B}^{\otimes m}\otimes\bar{\mathcal{B}}^{\otimes n}$ for all $m, n\geqq 0$, where $\bar{\mathcal{A}}$ is the C^*-algebra conjugate to $\mathcal{A}$ and $\mathcal{A}^{\otimes m}\otimes\bar{\mathcal{A}}^{\otimes n}$ is the projective C^*-tensor product of m copies of $\mathcal{A}$ and n copies of $\bar{\mathcal{A}}$. Finally in Section 4, we show that any completely positive map from $\mathcal{A}$ to a von Neumann