J. Math. Soc. Japan
Vol. 39, No. 2, 1987

Heegner points and the modular curve of prime level

By Benedict H. GROSS

(Received Nov. 30, 1985)

The purpose of this note is to show how Heegner points can be used to study the geometry of the modular curve $X=X_0(N)$ when N is prime. For example, we will show that the classical model for X in $\boldsymbol{P}^1\times\boldsymbol{P}^1$ given by the zeroes of the N^{th} modular polynomial has only ordinary double points as singularities. We will also consider a specific fibre system of elliptic curve over X when $N\equiv 3\ (\text{mod}\ 4)$ and relate the fibres over certain Heegner points to $\boldsymbol{Q}$-curves.

I wish to thank R. Rumely, J. Tate, and D. Zagier for suggesting some of the problems considered in this paper.

§1. Function theory.

Let N be a prime. The curve $Y=Y_0(N)$ is defined over $\boldsymbol{Q}$ and classifies elliptic curves with an N-isogeny. If F is any field of characteristic zero the points of Y rational over F correspond to diagrams

$$x=(\phi: E\rightarrow E'),$$

where E and E' are elliptic curves over F and ϕ is an F-rational (cyclic) isogeny of degree N. The complex points of Y may be identified with the Riemann surface $\mathfrak{H}/\Gamma_0(N)$ [**5**, §1].

The curve Y is non-singular, but is not complete. We denote its compactification $X=X_0(N)$; this is obtained by adjoining the two cusps ∞ and 0 which correspond to diagrams $(\phi: E\rightarrow E')$ of degenerate elliptic curves where the kernel of ϕ meets each geometric component of E [**1**, pp. 150-151]. We will call the points x of Y affine points of X; if x is a complex affine point we let τ be a pre-image of x in $\mathfrak{H}$ and $q=e^{2\pi i\tau}$.

The complex function field of X consists of the modular functions $f(\tau)$ for $\Gamma_0(N)$ which are meromorphic on the extended upper half-plane. A function f lies in the rational function field $\boldsymbol{Q}(X)$ if and only if the Fourier coefficients in its expansion at $\infty: f(\tau)=\sum a_n q^n$ are all rational numbers [**1**, p. 306]. The field $\boldsymbol{Q}(X)$ is known to be generated over $\boldsymbol{Q}$ by the functions