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# $\sqrt{\text{Morita}}$ theory

## —Formal ring laws and monoidal equivalences of categories of bimodules—

To the memory of Professor Akira Hattori

By Mitsuhiro TAKEUCHI

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### Introduction.

By an *equivalence data* between two categories $\mathcal{A}$, $\mathcal{B}$ we mean a 4-tuple $(\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, where $\boldsymbol{\Gamma}: \mathcal{A}\to\mathcal{B}$ and $\boldsymbol{\Delta}: \mathcal{B}\to\mathcal{A}$ are functors and $\boldsymbol{\gamma}: \boldsymbol{\Gamma\Delta}\simeq \boldsymbol{I}$, $\boldsymbol{\delta}: \boldsymbol{\Delta\Gamma}\simeq\boldsymbol{I}$ are isomorphisms of functors such that

$$\boldsymbol{\Delta\gamma}=\boldsymbol{\delta\Delta}, \qquad \boldsymbol{\gamma\Gamma}=\boldsymbol{\Gamma\delta}.$$

The Morita theory deals with equivalence data between left module categories ${}_R\mathcal{M}$, ${}_S\mathcal{M}$ for rings $R$, $S$. It is known that every equivalence data up to isomorphism is described in terms of some *Morita equivalence data* $({}_SP_R, {}_RQ_S, \alpha, \beta)$ with bimodule isomorphisms

$$\alpha : P\otimes_R Q\simeq S, \qquad \beta : Q\otimes_S P\simeq R$$

as follows: $\boldsymbol{\Gamma}$ takes $M\in{}_R\mathcal{M}$ to $P\otimes_R M\in{}_S\mathcal{M}$ and $\boldsymbol{\Delta}$ takes $N\in{}_S\mathcal{M}$ to $Q\otimes_S N\in{}_R\mathcal{M}$. The isomorphisms $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ come from $\alpha$, $\beta$ respectively.

When $\mathcal{A}$, $\mathcal{B}$ are *monoidal categories*, the 4-tuple $(\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ is called a *monoidal equivalence data* if in addition $\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}$ are *monoidal functors* and $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ are isomorphisms of monoidal functors. A basic example of a monoidal category is provided by ${}_R\mathcal{M}_R$ the category of all $R$-bimodules. For $R$-bimodules $M$, $N$, the tensor product $M\otimes_R N$ (of $M_R$ with ${}_RN$) has an $R$-bimodule structure (coming from ${}_RM$ and $N_R$). Together with unit $R$, this tensor product makes ${}_R\mathcal{M}_R$ into a monoidal category.

A natural question arises: *What happens if we consider monoidal equivalence data between bimodule monoidal categories* ${}_R\mathcal{M}_R$ *and* ${}_S\mathcal{M}_S$?

We begin with two simple examples of monoidal equivalence data. Let $({}_SP_R, {}_RQ_S, \alpha, \beta)$ be a Morita equivalence data as before. There is an associated

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