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Contact structures on twistor spaces

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Introduction.

A quaternionic Kähler manifold (M, g, H) is a Riemannian manifold (M, g) together with a coefficient bundle H of quaternions. The twistor space Z of (M, g, H), which is a complex manifold fibring over M, has a natural complex contact structure γ and a natural Einstein pseudo-Kähler metric $\bar{g}$, provided that (M, g) has non-zero scalar curvature (Salamon [8]). In this note we shall study the automorphism groups of these structures.

Let $\mathrm{Aut}(M, g, H)$ and $\mathrm{Aut}(Z, \gamma, \bar{g})$ denote the group of automorphisms of (M, g, H) and the one of isometric contact automorphisms of $(Z, \gamma, \bar{g})$ respectively. Then each element in $\mathrm{Aut}(M, g, H)$ can be lifted to an element in $\mathrm{Aut}(Z, \gamma, \bar{g})$ in a natural way. We show first that *the lifting homomorphism* $\mathrm{Aut}(M, g, H) \to \mathrm{Aut}(Z, \gamma, \bar{g})$ *is an isomorphism* (Theorem 3.1).

Let $\mathfrak{a}(Z, \gamma)$ and $\mathfrak{a}(Z, \gamma, \bar{g})$ be the Lie algebra of infinitesimal contact automorphisms of (Z, γ) and the one of infinitesimal isometric contact automorphisms of $(Z, \gamma, \bar{g})$ respectively. We prove next that $\mathfrak{a}(Z, \gamma)$ *is the complexification of* $\mathfrak{a}(Z, \gamma, \bar{g})$ (Corollary 2 to Theorem 3.2). This may be viewed as an analogue to the theorem of Matsushima to the effect that the Lie algebra of holomorphic vector fields on a compact Einstein Kähler manifold is the complexification of the Lie algebra of Killing vector fields.

Lastly we study a certain uniqueness of quaternionic Kähler structures. We prove: *Suppose that a compact complex contact manifold M admits a Kähler metric and has the vanishing first integral homology. Then a complex contact structure on M is unique up to automorphisms of M* (Theorem 1.7). Making use of this and previous results we show the following uniqueness: *Let compact quaternionic Kähler manifolds (M, g, H) and (M', g', H') with positive scalar curvatures have the same twistor space Z. Suppose that Z is a kählerian C-space of Boothby type* (see §1 for the definition). *Then (M, g, H) and (M', g', H') are equivalent to each other* (Theorem 4.2). Note that all the known examples of twistor spaces of compact quaternionic Kähler manifolds with positive scalar curvature are kählerian C-spaces of Boothby type.

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