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Provably recursive functions in fragments of Peano arithmetic

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§ 1. Introduction.

In this paper, we will make a proof-theoretic study of Paris-Harrington's independence results for Peano arithmetic [**10**]. First, we will give a characterization of provably recursive functions in fragments of Peano arithmetic. Then, we will analyze the combinatorial statements by Paris and Harrington, making use of our characterization.

Let PA be Peano's first-order arithmetic and PA* be the extension of PA obtained by adding all true Π_1-formulas as axioms. The fragments under consideration can be obtained by restricting induction formulas of the mathematical induction to formulas containing at most k quantifiers, for a given k.

Some basic facts on ordinal recursive functions and Wainer's hierarchy [**17**] will be presented in § 2. In § 3, we will prove our theorem on the characterization of provably recursive functions. By our theorem, the relation between α-ordinal recursive functions ($\alpha<\varepsilon_0$) and provably recursive functions in fragments of PA* will be clarified. Another characterization of these functions will be also stated in our theorem, in terms of the provability of some bounded formulas or some Δ_1-formulas in fragments of PA. We emphasize here that our theorem can be shown by using a purely proof-theoretic method.

This characterization enables us to analyze the combinatorial statements by Paris and Harrington which are shown to be independent of Peano arithmetic. This will be done in § 4, by making use of the estimation of rapidly growing functions associated with these combinatorial statements, due to Ketonen and Solovay [**3**]. Indeed, we can give an alternative proof of a result by Paris [**9**], in a proof-theoretic way. We will also mention explicitly how the provability or the unprovability of these statements depends on their representation in formal systems.

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