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Quasi-arithmetic means of continuous functions

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Introduction.

Let I be an interval, containing more than one point, of real numbers. A quasi-arithmetic mean with weight of two numbers a, b in I is defined as

(1) $$\phi^{-1}\{t\phi(a)+(1-t)\phi(b)\},$$

where ϕ is fixed as a strictly increasing or decreasing, real valued continuous function defined on I and the weight t is also fixed as a real number with $0<t<1$. This mean will be denoted by $N_{\phi,t}(a, b)$ throughout the paper.

This definition of a mean can be extended naturally to a mean of a continuous function, instead of two numbers, as follows. Let X be a compact Hausdorff space and let $C(X;I)$ be the space of all I-valued continuous functions on X. For a fixed strictly monotoneous continuous function ϕ on I as above and a fixed probability measure μ on X, a mean of a function f in $C(X;I)$ is defined to be

(2) $$\phi^{-1}\left\{\int_X \phi(f)d\mu\right\}.$$

In this paper this mean will be called a quasi-arithmetic mean with weight, simply a *QA-mean* and denoted by $M_{\phi,\mu}(f)$ for f in $C(X;I)$.

It is clear that a QA-mean $M=M_{\phi,\mu}$ is a continuous functional defined on $C(X;I)$ and has the following properties I), II) and (*), here we regard $C(X;I)$ as the space with the topology of uniform convergence and the usual order structure.

I) $M(a1_X)=a$ for all a in I,

where 1_X is the constant 1 function on X.

II) $M(f)\leqq M(g)$ if $f\leqq g$ in $C(X;I)$.

By Fubini's theorem we have the following equation (*), which will be called the *bisymmetry equation*, this terminology is taken from [2],

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