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# Behavior of modes of a class of processes with independent increments

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## 1. Introduction.

We consider a stochastic process $X_t$, $0 \leqq t < \infty$, with homogeneous independent increments of class $L$, starting at the origin. That is, $X_t$ is a process with homogeneous independent increments with characteristic function

$$E\exp(izX_t) = \exp\left[t\left(i\gamma z - 2^{-1}\sigma^2 z^2 + \int_{-\infty}^{\infty} g(z, x)x^{-1}k(x)dx\right)\right], \tag{1.1}$$

where

$$g(z, x) = e^{izx} - 1 - izx(1+x^2)^{-1}, \tag{1.2}$$

$\gamma$ real, $\sigma^2 \geqq 0$, $k(x)$ is non-negative, non-increasing on $(0, \infty)$ and non-positive, non-increasing on $(-\infty, 0)$, and

$$\int_{|x|<1} xk(x)dx + \int_{|x|>1} x^{-1}k(x)dx < \infty. \tag{1.3}$$

Yamazato [9] proves that the distribution of $X_t$ is unimodal for each $t$. If a distribution is unimodal, then either its mode is unique or the set of its modes is a closed interval. When $X_t$ has a unique mode, we denote it by $a(t)$. The purpose of the present paper is to study behavior of $a(t)$ as a function of $t$. In Section 2 we treat the case that $X_t$ is an increasing process (subordinator). Results and techniques of the paper [6] are employed. Asymptotic behavior of $a(t)$ as $t \to \infty$ is found when $k(x)$ is slowly varying at infinity. When $X_t$ is not an increasing process, behavior of $a(t)$ is hard to obtain except some asymptotic results. In Section 3 processes attracted to stable distributions are discussed. Other miscellaneous results are gathered in Section 4.

## 2. Increasing processes.

Assume that the process $X_t$ is increasing. This is equivalent to assuming

$$E\exp(izX_t) = \exp\left[t\left(i\gamma_0 z + \int_0^{\infty} (e^{izx} - 1)x^{-1}k(x)dx\right)\right], \tag{2.1}$$