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Kernels of Toeplitz operators

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1. Introduction.

Let U be the open unit disc in the complex plane and let ∂U be the boundary of U. If f is analytic in U and $\int_{-\pi}^{\pi}\log^{+}|f(re^{i\theta})|d\theta$ is bounded for $0\leqq r<1$, $f(e^{i\theta})$, which we define to be $\lim_{r\to 1}f(re^{i\theta})$, exists almost everywhere on ∂U. If

$$\lim_{r\to 1}\int_{-\pi}^{\pi}\log^{+}|f(re^{i\theta})|d\theta=\int_{-\pi}^{\pi}\log^{+}|f(e^{i\theta})|d\theta,$$

then f is said to be of the class N_{+}. The set of all boundary functions in N_{+} is again denoted by N_{+}. For $0<p\leqq\infty$, the Hardy space H^{p} is defined by $N_{+}\cap L^{p}$ where L^{p} denotes $L^{p}(d\theta)$. If $1\leqq p\leqq\infty$, it coincides with the space of functions in L^{p} whose Fourier coefficients with negative indices vanish. Put $H_{0}^{p}=\{f\in H^{p}: f(0)=0\}$. If $f\in L^{p}$ $(1<p<\infty)$ and $f\sim\sum_{n=-\infty}^{\infty}c_{n}e^{in\theta}$, then by a well-known theorem of M. Riesz (cf. [6, p. 54]) the series $\sum_{n=0}^{\infty}c_{n}e^{in\theta}$ is the Fourier series of a function Pf belonging to L^{p} (therefore, to H^{p}), and moreover $\|Pf\|_{p}\leqq A_{p}\|f\|_{p}$ where A_{p} is a constant depending only on p. Thus P is a bounded projection from L^{p} to H^{p}.

Let $\phi\in L^{\infty}$. We define the Toeplitz operator $\mathcal{T}_{\phi}$ on H^{p} by

$$\mathcal{T}_{\phi}f=P(\phi f).$$

Clearly $\mathcal{T}_{\phi}$ is a bounded operator with norm at most $A_{p}\|\phi\|_{\infty}$. We would like to define Toeplitz operators on H^{p} for $p=\infty$ or $0<p\leqq 1$. There we cannot use the projection P. Therefore for $0<p\leqq\infty$ we define the Toeplitz operator T_{ϕ} on H^{p} by

$$T_{\phi}f=\phi f+\bar{H}_{0}^{p}.$$

T_{ϕ} is a bounded operator with norm at most $\|\phi\|_{\infty}$ from H^{p} to $L^{p}/\bar{H}_{0}^{p}$. Denoting the kernel of T_{ϕ} by $\ker T_{\phi}$, we have clearly

$$\ker T_{\phi}=\ker\mathcal{T}_{\phi}$$

for $1<p<\infty$.

In § 4 of this paper, we determine under what conditions $\ker T_{\phi}$ is finite

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