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Some results on ordered fields

Dedicated to Professor Hirosi Nagao on his 60th birthday

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1. Introduction.

One of the purposes of this paper is to study the conjugacy of involutions (elements of order 2) in the automorphism group Aut(F) of an algebraically closed field F of characteristic 0. How many conjugacy classes of involutions are there in Aut(F)? This problem has its origin in the following question communicated to the author by Prof. H. Yabuki.

"*Let σ be an involutory automorphism of a field E. Let $x_1, \cdots, x_n$ be elements of E such that $x_1\sigma(x_1)+\cdots+x_n\sigma(x_n)=0$. Then can one say $x_1=\cdots=x_n=0$?*"

The answer is not always "yes" unless E is an algebraically closed field of characteristic 0. However in the case E is an algebraically closed field of characteristic 0, the answer is always "yes". This, being immediate from Artin-Schreier theory (see Proposition 2.6), led the author to study involutory automorphisms of an algebraically closed field. Since the characteristic of an algebraically closed field admitting an involutory automorphism is forced to be 0, we may restrict our attention to F. Let τ be an involutory automorphism of F. Then the fixed subfield of τ has codimension 2 and is real closed by virtue of Artin-Schreier theory (see [4] Satz 4). Indeed, this is a one-to-one correspondence between the involutions in Aut(F) and the subfields of F with codimension 2. Furthermore, two involutions are conjugate if and only if corresponding fixed subfields are isomorphic. Thus to study the conjugacy of involutions is equivalent to study the isomorphism problem of real closed fields. For this purpose we shall introduce two invariants of an ordered field K and show several properties concerning those invariants.

In the following, ord(Δ) denotes the order type of a totally ordered set Δ. Then our main results are as follows.

In Section 3, we introduce the term "*order-basis*" of K over its subfield L which is analogous to "transcendency basis", and we prove in Theorem A