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Virtual character modules of semisimple Lie groups and representations of Weyl groups

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Introduction.

Let G be a connected semisimple Lie group with finite centre and $\mathfrak{g}$ its Lie algebra. We call G acceptable if there exists a connected complex Lie group G_C with Lie algebra $\mathfrak{g}_C=\mathfrak{g}\otimes_R \boldsymbol{C}$ which has the following two properties. (1) The canonical injection from $\mathfrak{g}$ into $\mathfrak{g}_C$ can be lifted up to a homomorphism of G into G_C. (2) For a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}_C$, let ρ be half the sum of positive roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Then $\xi_\rho(\exp X)=\exp(\rho(X))$ $(X\in\mathfrak{h}_C)$ defines a character of H_C into $\boldsymbol{C}^*$.

We assume that G is acceptable throughout this paper.

For an irreducible quasi-simple representation π of G, we can associate π with an infinitesimal character $\lambda\in\mathfrak{h}_C^*$, where $\mathfrak{h}_C^*$ is the complex dual of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Also a distribution character $\Theta(\pi)$ of an irreducible quasi-simple representation π can be defined. We call $\Theta(\pi)$ an irreducible character of π which has an infinitesimal character λ. Let $V(\lambda)$ be the virtual character module of G whose element has an infinitesimal character λ.

In many papers, representations of the Weyl group $W=W(\mathfrak{h}_C)$ on the space $V(\lambda)$ are considered under the assumption that λ is regular and integral for G_C, i.e., λ is regular and is a differential of a character of H_C. G. Lustig and D. Vogan [**15**] considered W-module structure of $V(\lambda)$, using so-called "Springer representations". G. Zuckerman [**12**] also defined a representation of W on $V(\lambda)$, taking advantage of tensor products with finite dimensional representations of G. After his work, D. Barbasch and D. Vogan [**1**] restated his definition of the representation of W by means of "coherent continuation" and determined the W-module structure in the case that G is a connected reductive group with all the Cartan subgroups connected and that G has a compact Cartan subgroup. On the other hand, representations of the Weyl group W on the space of so-called Goldie rank polynomials are considered by A. Joseph [**10**], D.R. King [**11**] and others. It seems that these representations on the space of Goldie rank polynomials or the character polynomials can be realized as subrepresentations of the representation on a virtual character module $V(\lambda)$.