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Asymptotic properties of asymptotically homogeneous diffusion processes on a compact manifold

By Osamu TAKEYAMA

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§ 1. Introduction.

The purpose of this paper is to formulate a class of time inhomogeneous diffusion processes and to investigate the asymptotic properties of such processes. Let $\{\xi(t), P_{s,x}\}$ be a time inhomogeneous diffusion process on a manifold M generated by a smooth differential operator

$$L_t=\frac{1}{2}a^{ij}(t, x)\frac{\partial^2}{\partial x^i\partial x^j}+b^i(t, x)\frac{\partial}{\partial x^i}$$

and let $\{\lambda(t), P_x\}$ be a homogeneous diffusion process on M generated by a smooth differential operator

$$L=\frac{1}{2}a^{ij}(x)\frac{\partial^2}{\partial x^i\partial x^j}+b^i(x)\frac{\partial}{\partial x^i}.$$

(Throughout this paper we use the usual summation convention.) Here, $P_{s,x}$ is the probability law governing sample paths $\xi(t)$, $t\geqq s$, starting at x at time s and P_x is that of $\lambda(t)$, $t\geqq 0$, starting at x at time 0. If $L_t\phi\to L\phi$ uniformly on any compact set on M as $t\to\infty$ for every smooth function ϕ on M, we shall call $\{\xi(t), P_{s,x}\}$ *asymptotically homogeneous* with the limiting homogeneous process $\{\lambda(t), P_x\}$.

Such a situation was studied by Bhattacharya and Ramasubramanian [2]. They showed that if $\{\lambda(t), P_x\}$ is positively recurrent with the invariant probability measure m, then, under additional assumptions on the process, the law of the shifted process $t\mapsto\xi_s^+(t)=\xi(t+s)$ under $P_{0,x}$ converges to that of $t\mapsto\lambda(t)$ under $P_m=\int_M P_x m(dx)$ as $s\to\infty$ for every $x\in M$. Conversely, if this convergence holds then $\{\xi(t), P_{s,x}\}$ must be asymptotically homogeneous. Thus we may expect that if an inhomogeneous diffusion $\{\xi(t), P_{s,x}\}$ is asymptotically homogeneous with the limiting homogeneous diffusion process $\{\lambda(t), P_x\}$, the asymptotic properties of the process $\xi(t)$ can be stated in terms of the process $\lambda(t)$. In this paper, we discuss the asymptotics of the occupation distribution (the empirical distribution) for an asymptotically homogeneous diffusion process.

In § 2, we obtain some preliminary general results on inhomogeneous diffusion processes. If a general inhomogeneous diffusion operator L_t is smooth, L_t-diffusion