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Stiefel-Whitney homology classes and homotopy type of Euler spaces

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1. Introduction.

In this paper, we construct Euler spaces in fixed homotopy types such that the Stiefel-Whitney homology classes are equal to given homology elements. As a byproduct, we obtain counterexamples to Halperin's conjecture (Fulton-MacPherson [**4**]).

Let X be a locally compact n-dimensional polyhedron. For a point x in X, let $\chi(X, X-x)$ denote the Euler number of the pair $(X, X-x)$. The polyhedron X is called an *integral Euler space* (resp. *mod 2 Euler space*) if for each x in X, $\chi(X, X-x)=(-1)^n$ (resp. $\chi(X, X-x)\equiv 1 \pmod 2$) (Halperin and Toledo [**6**]). Sullivan [**9**] has shown that complex analytic spaces (resp. real analytic spaces) are integral Euler spaces (resp. mod 2 Euler spaces).

Let K' denote the barycentric subdivision of a triangulation K of a polyhedron X. If X is a mod 2 Euler space, the sum of all k-simplexes in K' is a mod 2 cycle and defines an element $s_k(X)$ in $H_k(X;\boldsymbol{Z}_2)$ (cf. [**6**]). Note that, if X is not compact, we consider the homology of infinite chains. The element $s_k(X)$ is called the *k-th Stiefel-Whitney homology class* of X. If X is connected and compact, $s_0(X)$ is the mod 2 reduction of the Euler number $\chi(X)$, where we identify $H_0(X;\boldsymbol{Z}_2)$ with $\boldsymbol{Z}_2$. If X is a smooth manifold, PL-manifold, or $\boldsymbol{Z}_2$-homology manifold, the class $s_k(X)$ is known to be equal to the Poincaré dual of the Stiefel-Whitney cohomology class $w^{n-k}(X)$ (Cheeger [**3**], Halperin-Toledo [**6**], Taylor [**10**], Blanton-McCrory [**2**], Veljan [**11**], Matsui [**8**]). Consequently, for such spaces, the Stiefel-Whitney homology classes $s_*(X)$ are homotopy type invariant. For further properties of Stiefel-Whitney homology classes, see [**1**], [**7**].

A polyhedron X is called purely n-dimensional if the union of all n-simplexes in a triangulation of X is dense in X. We have the following concerning mod 2 Euler spaces:

THEOREM 1. *Let X be a purely n-dimensional* mod 2 *Euler space and let a_i, for $i=1, 2, \cdots, n-1$, be elements in $H_i(X;\boldsymbol{Z}_2)$. Then there exist a purely n-dimensional* mod 2 *Euler space Y and a homotopy equivalence $h: X\to Y$ such that $h_*(a_i)=s_i(Y)$ for $i=1, 2, \cdots, n-1$ and $h_*s_n(X)=s_n(Y)$.*