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Rough isometries, and combinatorial approximations of geometries of non-compact riemannian manifolds

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1. Introduction.

Among the most elementary relations in mathematics are the equivalence relations. Especially in geometry, equivalence relations between manifolds are usually defined by the existences of maps satisfying some suitable conditions. An isometry is the most fundamental map of this kind and defines the equivalence relation of isometry between riemannian manifolds. Another kind of maps which defines an equivalence relation between riemannian manifolds is the quasi-isometry. Now suppose that X and Y are riemannian manifolds. A diffeomorphism of X onto Y is called a *quasi-isometry* if there is a constant $a\geqq 1$ such that

$$a^{-1}|\xi|\leqq|d\varphi(\xi)|\leqq a|\xi| \qquad \text{for all} \quad \xi\in TX. \tag{1.1}$$

When there is a quasi-isometry from X onto Y, we say that X is *quasi-isometric* to Y: Obviously to be quasi-isometric is an equivalence relation. Another map belonging to a broader class is a pseudo-isometry introduced by Mostow [**15**] for studying discontinuous subgroups of semi-simple Lie groups. A *pseudo-isometry* of X into Y is a continuous map satisfying

$$a^{-1}d(x_1, x_2)-b\leqq d(\varphi(x_1), \varphi(x_2))\leqq ad(x_1, x_2) \qquad \text{for all} \quad x_1, x_2\in X \tag{1.2}$$

with suitable constants $a\geqq 1$ and $b\geqq 0$, where d denotes the distance functions of X and Y induced from their riemannian structures. It is evident that a quasi-isometry is a pseudo-isometry. But, as is expected from the observation that (1.2) is not symmetric, the existence of a pseudo-isometry does not define an equivalence relation; in fact there exists a pair of complete riemannian manifolds X and Y such that there is a pseudo-isometry from X onto Y but there are no pseudo-isometries of Y into X.

Now we introduce another kind of maps called rough isometries which satisfy a condition weaker than (1.1) and (1.2). Let X be a metric space. For a point x in X, $B_r(x)$ denotes the open r-ball around x: Moreover for a subset