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On a conformally invariant functional of the space of Riemannian metrics

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Introduction.

Let $\mathcal{M}(M)$ be the space of all C^∞ Riemannian metrics on a compact n-dimensional manifold M, and $\nu: \mathcal{M}(M) \to \boldsymbol{R}$ be a functional of $\mathcal{M}$ defined by $\nu(g)=(2/n)\int_M |W|^{n/2}dv$, where W is the Weyl conformal curvature tensor. Our main subject in this paper is to determine $\inf\{\nu(g); g\in\mathcal{M}\}$, which will be denoted by $\nu(M)$. A little consideration shows that $\nu(M)>0$ if some Pontrjagin number of M is not zero. Thus, in general, $\nu(M)$ is a nontrivial invariant of a manifold.

In § 2, we shall show two general properties of $\nu(M)$. One is that $\nu(M)=0$ for the total space M of a principal circle bundle (Theorem 2.1). This provides examples of M for which $\nu(M)=0$ but which has no conformally flat metric. The other is an inequality for connected sum; $\nu(M_1 \# M_2)\leqq \nu(M_1)+\nu(M_2)$ (Theorem 2.2). This is useful for computing $\nu(M)$ for certain M.

However, to determine $\nu(M)$ for general M seems to be not so easy. Even for $S^2\times S^2$, $\nu(S^2\times S^2)$ is not known (to the author). We want to show that the standard Einstein metric g_0 of $S^2\times S^2$ is a candidate at which ν takes a minimum, if $\nu: \mathcal{M}(S^2\times S^2)\to\boldsymbol{R}$ has a minimum. In fact, g_0 is a minimum point of ν restricted to Kähler metrics (Proposition 1.4). Moreover, we shall prove that g_0 is a strictly stable critical point of the functional ν (cf. Definition 4.1 and Theorem4 .2).

In the course of proof of stability of $g_0\in\mathcal{M}(S^2\times S^2)$, we establish the first and the second variational formulas of $\nu: \mathcal{M}(M)\to\boldsymbol{R}$ for 4 dimensional M (Propositions 3.1 and 3.7; The first variational formula has already appeared in [**2**]). From these formulas, we can also see that other than conformally flat metrics, Einstein metrics are critical points of the functional ν, and Ricci flat metrics are stable critical points of ν.

§ 1. Preliminary definitions and remarks.

Throughout this paper, M denotes a compact C^∞ manifold of dimension n, and $\mathcal{M}(M)$ denotes the space of C^∞ Riemannian metrics on M. For $g\in\mathcal{M}(M)$,