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Helical minimal imbeddings of order 4 into spheres

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§ 0. Introduction.

Let $f: M\to\bar{M}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\bar{M}$. If for each geodesic γ of M the curve $f\circ\gamma$ in $\bar{M}$ is of osculating order d and has constant curvatures which are independent of the choice of γ, then f is called a *helical immersion of order d*. In this paper we shall study helical minimal immersions of order 4 into a unit sphere $S(1)$.

Besse [**1**] showed that a strongly harmonic manifold admits a helical minimal immersion into a sphere. As is well-known, making use of eigenfunctions of the Laplace operator, we obtain minimal immersions of compact rank one symmetric spaces into spheres (cf. [**14**]). Similarly we have the α-th standard minimal immersions of strongly harmonic manifolds into spheres. Let m_α be the multiplicity of the α-th eigenvalue of the Laplace operator and ϕ_i $(i=1, \cdots, m_\alpha)$ an orthonormal base for its eigenspace. Then we define Φ_α by $\Phi_\alpha(x)=(\phi_1(x), \cdots, \phi_{m_\alpha}(x))\in \boldsymbol{R}^{m_\alpha}$. If we change homothetically the metric on the strongly harmonic manifold, then Φ_α becomes a helical minimal immersion into a hypersphere of $\boldsymbol{R}^{m_\alpha}$. We call Φ_α the *α-th standard minimal immersion* of strongly harmonic manifolds. Tsukada [**13**] proved that if $f: M\to S(1)$ is a helical minimal immersion of a strongly harmonic manifold M, then f is equivalent to some Φ_α, that is, $f=\Psi\circ\Phi_\alpha$ with some isometry Ψ of $S(1)$.

Let $f: M\to S(1)$ be a helical minimal imbedding of a compact n-dimensional Riemannian manifold M. If the order d of f is equal to 1, then f is totally geodesic. In the case $d=2$, Little [**5**] and the author [**8**] showed that M is isometric to one of real projective space $\boldsymbol{RP}^n$, complex projective space $\boldsymbol{CP}^m$ $(n=2m)$, quaternion projective space $\boldsymbol{QP}^m$ $(n=4m)$ and Cayley projective space $\boldsymbol{CayP}^2$ $(n=16)$ with canonical metrics and f is equivalent to Φ_1. If $d=3$, then M is isometric to S^n and $f\approx\Phi_3$. This result was given by Nakagawa [**6**] (see also [**10**], [**11**]). The case $d=4$ was studied in [**11**] and proved that M is isometric to one of projective spaces $\boldsymbol{RP}^n$, $\boldsymbol{CP}^m$, $\boldsymbol{QP}^m$ and $\boldsymbol{CayP}^2$ under the condition that $a=\langle\dot\gamma(0), \dot\gamma(L)\rangle>0$ for any unit speed geodesic γ where L is the diameter of M and $\langle\,,\rangle$ denotes the inner product of the Euclidean space in which $S(1)$ is naturally imbedded (also was proved that $f\approx\Phi_2$). Furthermore if $d=5$, then