J. Math. Soc. Japan
Vol. 37, No. 2, 1985

Propagation of chaos for the Burgers equation

By Hirofumi OSADA and Shinichi KOTANI

(Received May 14, 1984)

§ 0. Introduction.

In [5] H. P. McKean considered systems of many particles obeying stochastic differential equations:

$$dX_i^n=\frac{1}{n-1}\sum_{j\neq i}^{n} e(X_i^n, X_j^n)dB_i+\frac{1}{n-1}\sum_{j\neq i}^{n} f(X_i^n, X_j^n)dt \qquad (i=1, 2, \cdots, n). \tag{0.1}$$

Under the conditions of smoothness and boundedness of the coefficients, he proved that if the initial values $X_i^n(0)$ are i. i. d. random variables then any fixed finite particles converge to independent copies of a one-dimensional diffusion process determined by an equation

$$dX(t)=e[X(t), p_t]dB(t)+f[X(t), p_t]dt, \tag{0.2}$$

where $e[x, \mu]=\int_R e(x, y)\mu(dy)$ and p_t is a distribution of $X(t)$. A point of the proof is in studying the processes $\{X_i^n(t)\}$ in the infinite product probability space $\prod_{i=1}^{\infty}\{\mu_i, P_i\}$ of identically and independently distributed initial distribution and Brownian motions, and applying Hewitt-Savage's 0-1 law to these diffusion processes on this probability space. There is another approach to this problem employed by H. Tanaka and A. S. Sznitman [9], [8]. They discussed probability measure-valued processes $(1/n)\sum_{i=1}^{n}\delta_{X_i^n(t)}$ from the point of view of a martingale problem. In these arguments the smoothness of the coefficients e, f is crucial. However an interesting case of $e(x)=1$ and $f(x)=(\lambda/2)\,\delta(x)$ is excluded. In this case the expected limit process satisfies

$$dZ(t)=dB(t)+\frac{\lambda}{2}p_t(Z(t))dt, \tag{0.3}$$

and $p_t(x)$ is a solution of the Burgers equation

$$\frac{\partial}{\partial t}p=\frac{1}{2}\nabla^2 p-\frac{\lambda}{2}\nabla p^2 \qquad \left(\nabla=\frac{\partial}{\partial x}\right). \tag{0.4}$$

(0.4) is uniquely solvable for any initial distribution in the following way:

$$p_t(x)=-\frac{1}{\lambda}\cdot\frac{\partial}{\partial x}\left\{\log\int_R g_t(x-y)e^{-\lambda\int_{-\infty}^{y}\nu(dz)}dy\right\}, \qquad g_t(x)=\frac{1}{(2\pi t)^{1/2}}e^{-x^2/2}.$$