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Calculation of the class numbers and fundamental units of abelian extensions over imaginary quadratic fields from approximate values of elliptic units

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§ 0. Introduction.

0.1. Any number field we consider is a finite extension of the rational number field $\boldsymbol{Q}$ in the complex number field $\boldsymbol{C}$.

Let L/F be an abelian extension of number fields. For L, denote its class number by h and its group of units by E. We restrict our study to either of the following cases:

CASE 1. *$F=\boldsymbol{Q}$ and L is contained in the real number field $\boldsymbol{R}$.*

CASE 2. *F is an imaginary quadratic number field.*

In this paper, we give a general procedure to calculate h and to find together fundamental units of L. We first connect h to a finite index subgroup $\boldsymbol{E}$ of E by an index formula of the form $h=c(E:\boldsymbol{E})$ (Theorem 2 below). Hence c ($\in\boldsymbol{Q}$) is rather easy to know and $\boldsymbol{E}$ is generated by cyclotomic (Case 1) or so called *elliptic* (Case 2) units. The process to decide $(E:\boldsymbol{E})$ starts from the generators of $\boldsymbol{E}$, and ends at a free basis of E (Algorithm 4 below). Thus h and, at a time, fundamental units are obtained. To make the process effective, an upper bound of $(E:\boldsymbol{E})$ should be known beforehand. So we majorize $(E:\boldsymbol{E})$ by using the generators of $\boldsymbol{E}$ (Theorem 3).

Our method will be computer implementable, though we do not discuss it in detail. What we emphasize is that the classical (explicit) theory of cyclotomic fields or complex multiplication offers us a new general way of calculating h and E as above.

We are mainly interested in Case 2. Because, in Case 1, our formula for h is that of Leopoldt [**14**] and the principle of calculation is the same as in Gras-Gras [**6**]. Investigating Case 1 together, we improve Gras-Gras's method itself. In Case 2, an analogy of Leopoldt's formula has been given by Schertz [**24**, I] or Gillard-Robert [**5**], which, however, has not taken Gras-Gras's method into account. This tempts us to prove a more appropriate analogy of Leopoldt's formula as in Theorem 2.