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Pointwise multipliers for functions of bounded mean oscillation

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1. Introduction.

The purpose of this paper is to characterize the set of pointwise multipliers on $bmo_{\phi}(\boldsymbol{R}^n)$, which is the function space defined using the mean oscillation and a growth function ϕ.

Janson [2] has characterized pointwise multipliers on $bmo_{\phi}(\boldsymbol{T}^n)$ on the n-dimensional torus $\boldsymbol{T}^n$. We extend his result to the case of the n-dimensional Euclidean space $\boldsymbol{R}^n$.

To define $bmo_{\phi}(\boldsymbol{R}^n)$, let $I(a, r)$ be the cube $\{x\in\boldsymbol{R}^n ; |x_i-a_i|\leqq r/2, i=1, 2, \cdots, n\}$ whose edges have length r and are parallel to the coordinate axes. For a cube I, we denote by $|I|$ the Lebesgue measure of I, by $M(f, I)$ or f_I the mean value of a function f on I, i.e. $|I|^{-1}\int_I f(x)dx$, and by $MO(f, I)$ the mean oscillation of f on I, i.e. $|I|^{-1}\int_I |f(x)-f_I|dx$.

We now define

$$bmo_{\phi}(\boldsymbol{R}^n)=\left\{f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n) \ ; \ \sup_{I(a,r)} \frac{MO(f, I(a, r))}{\phi(r)}<+\infty\right\},$$

where ϕ is assumed to be a positive non-decreasing function on $\boldsymbol{R}_+=(0, \infty)$. Such a function is called a growth function. If two growth functions ϕ_1 and ϕ_2 are equivalent ($\phi_1\sim\phi_2$) i.e. there is a constant $C>0$ such that $C^{-1}\phi_1(r)\leqq\phi_2(r)\leqq C\phi_1(r)$, then $bmo_{\phi_1}(\boldsymbol{R}^n)=bmo_{\phi_2}(\boldsymbol{R}^n)$.

A function g on $\boldsymbol{R}^n$ is called a pointwise multiplier on $bmo_{\phi}(\boldsymbol{R}^n)$, if the pointwise multiplication fg belongs to $bmo_{\phi}(\boldsymbol{R}^n)$ for all f belonging to $bmo_{\phi}(\boldsymbol{R}^n)$.

Janson's characterization is the following. If ϕ is a growth function and $\phi(r)/r$ is almost decreasing, then a function g is a pointwise multiplier on $bmo_{\phi}(\boldsymbol{T}^n)$ if and only if g belongs to $bmo_{\psi}(\boldsymbol{T}^n)\cap L^{\infty}(\boldsymbol{T}^n)$ where $\psi(r)=\phi(r)\Big/\int_r^1\phi(t)t^{-1}dt$. (A positive function $h(t)$ is said to be almost decreasing if there

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