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Steepest descent and differential equations

In memory of my teacher, H. S. Wall

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1. Introduction.

This note is a report on some phenomena suggested by numerical solution of boundary value problems. Two conditions are given. If both hold for a given problem then continuous constrained steepest descent converges to a solution.

Suppose that each of H, K and S is a real Hilbert space, $F: H \to K$, $B: H \to S$ and each of F and B has a locally Lipschitz derivative.

Consider the problem of *constructively* identifying $u \in H$ such that

$$F(u)=0, \qquad B(u)=0. \tag{1}$$

Many boundary value problems in differential equations — ordinary, partial, functional — can be cast as such problems where $F(u)=0$ represents a differential equation and $B(u)=0$ represents boundary conditions.

Denote by P the function on H so that if $x \in H$ then $P(x)$ is the orthogonal projection of H onto $N(B'(x))$, the nullspace of $B'(x)$. It is assumed throughout that P is locally Lipschitz.

Define ϕ on H so that if $x \in H$ then

$$\phi(x)=\|F(x)\|^2/2, \qquad x \in H$$

and denote by $\nabla_B \phi$ the function defined on H so that

$$(\nabla_B \phi)(x)=P(x)(\nabla \phi)(x), \qquad x \in H$$

where $\nabla \phi$ is the gradient function for ϕ. $\nabla_B \phi$ is called the B-gradient of ϕ. The following is intended to justify this terminology:

LEMMA 1. *Suppose $x \in H$ and α_x is the function with domain $N(B'(x))$ so that*

$$\alpha_x(k)=\phi(x+k), \qquad k \in N(B'(x)).$$

Then $(\nabla_B \phi)(x)=(\nabla \alpha_x)(0)$.

LEMMA 2. *If $x \in H$ there is a unique function z from $[0, \infty)$ to H so that*

$$z(0)=x, \qquad z'(t)=-(\nabla_B \phi)(z(t)), \quad t \geqq 0. \tag{2}$$