J. Math. Soc. Japan
Vol. 36, No. 4, 1984

On Shimura's elliptic curve over $\boldsymbol{Q}(\sqrt{29})$

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(Received Dec. 5, 1983)

Let k be the real quadratic field $\boldsymbol{Q}(\sqrt{29})$. Then the class number of k is 1 and $\varepsilon=(5+\sqrt{29})/2$ is a fundamental unit of k. Let E_0 be an elliptic curve over k defined by the equation:

$$y^2+xy+\varepsilon^2 y=x^3.$$

Let B be the elliptic curve over k which is obtained from the space $S_2\left(\Gamma_0(29), \left(\frac{}{29}\right)\right)$ of cusp forms of "Neben"-type of weight 2 (see Shimura [**4**, §7.5, §7.7]). It is conjectured that B is isogenous to E_0 over k (see Serre [**3**, p. 323] and Shimura [**5**, p. 184]). It will be shown here that this is so, by reducing the problem to the solution of a certain diophantine equation over k.

§1. Let σ be the non-trivial automorphism of k and O_k the integer ring of k. Let E be an elliptic curve over k. For a natural number n, we denote by E_n the group of elements x of $E(\bar{k})$ with $nx=0$.

THEOREM. *Let E be an elliptic curve over k. Assume that E satisfies the following conditions:*

(i) *E has everywhere good reduction over k.*

(ii) *E has an isogeny onto E^σ over k whose degree is prime to* 6.

(iii) *E has a k-rational point of order* 3.

(iv) *$[k(E_2):k]$ is divisible by* 2.

(v) *$[k(E_3):k]$ is divisible by* 3.

Then E is k-isomorphic to either E_0 or E_0^σ.

REMARK. The condition (ii) of Theorem implies that $k(E_2)$ and $k(E_3)$ are Galois over $\boldsymbol{Q}$.

COROLLARY. *Shimura's elliptic curve B is isogenous to E_0 over k.*

PROOF OF COROLLARY. By Casselman [**1**], B has everywhere good reduction. It is known that B has an isogeny onto B^σ of degree 5. Since the number of the $\boldsymbol{F}_{p^2}$-rational points of the reduction of B at $p=3$ is $1-(2p+a_p^2)+p^2=9$ ($a_p=-\sqrt{-5}$, cf. Yamauchi [**6**]), we have $k(B_2)\neq k$. By (i), $k(B_2)/k$ is unramified

This research was partially supported by Grant-in-Aid for Scientific Research (No. 59540008), Ministry of Education.