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On the stability of minimal surfaces in $\boldsymbol{R}^3$

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§ 0. Introduction.

Minimal surfaces are exactly the critical points of area functional for all variations which keep their boundary values fixed. But they do not necessarily provide relative minima of area. When a minimal surface corresponds to a relative minimum of area for all such variations, we say it is stable, otherwise unstable.

In this paper we shall give sufficient conditions for the stability and the instability of minimal surfaces in the Euclidean space $\boldsymbol{R}^3$.

Let D be a plane domain with compact closure $\bar{D}$, whose boundary ∂D is a finite union of piecewise C^∞ curves. Let $\mathfrak{X} : \bar{D} \to \boldsymbol{R}^3$ be a regular (i.e. immersed) minimal surface. And denote by $\mathfrak{G} : \bar{D} \to S = \{(x^1, x^2, x^3) \in \boldsymbol{R}^3 ; (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$ the Gauss map of $\mathfrak{X}$. Barbosa and do Carmo [1] gave a sufficient condition for $\mathfrak{X}$ to be stable:

THEOREM (Barbosa and do Carmo [1]). *If the area of $\mathfrak{G}(\bar{D})$ (as a point set on S) is smaller than 2π, then $\mathfrak{X}$ is stable.*

This estimate is sharp in the following sense: There are examples of unstable minimal surfaces whose Gaussian image has area larger than 2π and as close to 2π as one pleases.

Now what can we say about the stability of minimal surfaces satisfying the condition that the area of $\mathfrak{G}(\bar{D})$ is exactly 2π? Our purpose in this paper is to answer this question.

Except the case that $\mathfrak{G}$ is a branched covering of a hemisphere H of S (i.e. $\mathfrak{G}(\bar{D})$ coincides with H and $\mathfrak{G}(\partial D) = \partial H$), minimal surface $\mathfrak{X}$ is always stable (Theorem 1 and Theorem 2).

In the case excepted above, let f and g be the factors of the Weierstrass representation of $\mathfrak{X}$ (cf. § 3). By a suitable rotation of the surface in $\boldsymbol{R}^3$, we may assume that $\mathfrak{G}(\bar{D})$ coincides with the lower hemisphere of $S : H^- = \{(x^1, x^2, x^3) \in S ; x^3 \leqq 0\}$. In this situation, g is a holomorphic function of D onto $D_0 = \{w \in \boldsymbol{C} ; |w| < 1\}$. Here we define a function F in D_0 as follows: