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A pinching problem for symmetric spaces of rank one

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§ 0. Introduction.

A main problem in Riemannian geometry is to investigate the influences of geometrical quantities of complete Riemannian manifolds on the topology. The pioneering work for this is the well known sphere theorem due to Rauch [12] which was improved by Klingenberg [9]. Take $M=S^n$ with the constant sectional curvature equal to 1. The theorem states that if $\bar{M}$ is a complete simply connected n-manifold with the sectional curvature $K_{\bar{M}}$, $\frac{1}{4}<K_{\bar{M}}\leqq 1$, then $\bar{M}$ is homeomorphic to S^n. A stronger assumption for curvature implies that $\bar{M}$ must be diffeomorphic to S^n ([6], [14], [16]).

Cheeger [3] defines another notion of pinching. Let M, $\bar{M}$ be compact Riemannian manifolds of $\dim M=\dim \bar{M}=n$ and $m\in M$, $\bar{m}\in\bar{M}$. Let $I: M_m\to\bar{M}_{\bar{m}}$ be a linear isometry between the tangent spaces. For a geodesic γ emanating from m, let $\bar{\gamma}$ denote the geodesic emanating from $\bar{m}$ such that $\bar{\gamma}'(0)=I(\gamma'(0))$, and P_γ the parallel translation along γ. Set $I_\gamma:=P_{\bar{\gamma}}\circ I\circ P_{\gamma^{-1}}$. I_γ induces an isomorphism on tensor spaces. We denote by R_M the curvature tensor of M and by $L(\ )$ the length of curves. Now set:

$$\tilde{\rho}(M, \bar{M}) := \inf_{m,\bar{m},I} [\sup\{\|R_M-I_\gamma{}^{-1}(R_{\bar{M}})\| \ ;\ L(\gamma)\leqq 2\operatorname{diam}(M)\}] .$$

Let M be a simply connected compact rank one symmetric space (henceforth SCROSS). One of his results states that there exists an $\varepsilon>0$ such that if a compact simply connected manifold $\bar{M}$ is ε-close to M with respect to $\tilde{\rho}$, then $\bar{M}$ is piecewise linearly homeomorphic to M.

The main purpose of this paper is to consider diameter or volume-pinching for SCROSSes using a somewhat weaker one than $\tilde{\rho}$, as well as to strengthen the topological conclusion to diffeomorphism. For $m\in M$, we denote by $\mathfrak{S}_m$ the compact domain in M_m bounded by the tangent cut locus of m: $\mathfrak{S}_m:=\{v\in M_m\ ;\ d(\exp_m v, m)=\|v\|\}$, and by $U\mathfrak{S}_m$ the set of all unit tangent vectors on $\mathfrak{S}_m$. We define our pinching numbers by

DEFINITION.

$$\rho_0(M, \bar{M})= \inf_{m,\bar{m},I} [\sup\{|\,\|d\exp_{\bar{m}} I(v)\|-\|d\exp_m(v)\|\,|\ ;\ v\in U\mathfrak{S}_m\}] ,$$