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The asymptotic formulas for the number of bound states in the strong coupling limit

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Introduction.

The problem we want to discuss in the present paper is that of the asymptotic number of bound state energies (negative eigenvalues) of the Schrödinger operator $-\Delta+\lambda V$, $\lambda>0$, in the strong coupling limit $\lambda\rightarrow\infty$. This problem has been already discussed by many authors. See, for example, Birman-Borzov [**5**], Kac [**10**], Lieb [**13**], Martin [**14**], Reed-Simon [**17**], Rosenbljum [**19**] and the references quoted there. Roughly speaking, in the case of 3-dimensional space $\boldsymbol{R}_x^3$, the result obtained by these authors can be formulated as follows: Assume that $V(x)$ is real and $V(x)\in L^{3/2}(\boldsymbol{R}_x^3)$, $L^p(\boldsymbol{R}_x^3)$ being the Lebesgue space, and denote by $N(\lambda)$ the number of bound state energies of $-\Delta+\lambda V$. Then $N(\lambda)$ obeys the asymptotic formula

$$N(\lambda)=(6\pi^2)^{-1}\int|V_-(x)|^{3/2}dx\,\lambda^{3/2}(1+o(1)),\qquad \lambda\rightarrow\infty,$$

where $V_-(x)$ denotes the attractive part of $V(x)$; $V_-(x)=\min(0, V(x))$, and the integration is taken over the whole space $\boldsymbol{R}_x^3$. (Here and in what follows, integration with no domain attached is taken over the whole space.) For the proof, [**5**], [**14**] and [**19**] use the min-max principle combined with a technique of Dirichlet-Neumann bracketing, while [**10**], [**13**] and [**17**] use the Feynman-Kac formula. The aim of the present paper is to derive a similar asymptotic formula with the improved remainder estimate $O(\lambda^{-1/2})$ under rather restrictive assumptions on $V(x)$.

We shall formulate the main theorem precisely. We work in the 3-dimensional space and consider only a class of attractive potentials, so it is convenient in the discussion below to write the Schrödinger operator as $-\Delta-\lambda V$, $V>0$, without using the standard notation $-\Delta+\lambda V$. Furthermore, the class of potentials we consider admits finite singularities. For brevity, we confine ourselves to potentials having singularities at the origin only. Such potentials are important in a physical application.

First, we make the assumptions on $V(x)$, which specify the behavior of $V(x)$ as $|x|\rightarrow 0$ and as $|x|\rightarrow\infty$. To describe these assumptions, we follow the standard