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Spherical t-designs which are orbits of finite groups

Dedicated to Professor Hirosi Nagao on the occasion
of his 60-th birthday

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Introduction.

A spherical t-design in $\boldsymbol{R}^d$ is a finite nonempty subset X in the unit sphere $\Omega_d=\{(x_1, \cdots, x_d)\in\boldsymbol{R}^d \mid x_1^2+\cdots+x_d^2=1\}$ such that

(0.1) $$\frac{1}{|\Omega_d|}\int_{\Omega_d} f(\boldsymbol{x})d\omega(\boldsymbol{x})=\frac{1}{|X|}\sum_{\boldsymbol{x}\in X} f(\boldsymbol{x})$$

for all polynomials $f(\boldsymbol{x})=f(x_1, x_2, \cdots, x_d)$ of degree $\leqq t$. The condition (0.1) is equivalent to the following condition:

(0.2) $$\sum_{\boldsymbol{x}\in X} f(\boldsymbol{x})=0$$

for all homogeneous harmonic polynomials $f(\boldsymbol{x})=f(x_1, \cdots, x_d)$ of degrees $1, 2, \cdots, t$.

The reader is referred to Delsarte-Goethals-Seidel [6] for the basic properties of spherical t-designs. In this paper, we study spherical t-designs X which are obtained from finite subgroups G of the real orthogonal group $O(d)$ in such a way that

$$X:=\boldsymbol{x}^G:=\{\boldsymbol{x}^g \mid g\in G\}\subset\Omega_d$$

for some $\boldsymbol{x}\in\Omega_d$. (Namely, X is a spherical t-design which is obtained as an orbit of a finite group G in $O(d)$.)

Let G be a finite subgroup of the real orthogonal group $O(d)$ acting on $\boldsymbol{R}^d$ and on Ω_d. Let ρ_i ($i=0, 1, 2, \cdots$) be the i-th spherical representation of $O(d)$, i.e., the representation of $O(d)$ on the space of homogeneous harmonic polynomials of degree i. So

$$\dim\rho_i=\binom{d+i-1}{i}-\binom{d+i-3}{i-2}.$$

In [1] the following theorem was proved:

THEOREM A (Bannai [1, Theorem 1]). (i) *Let G be a finite subgroup of*

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