J. Math. Soc. Japan
Vol. 36, No. 1, 1984

The OE-property of group automorphisms

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(Received Feb. 16, 1982)
(Revised Jan. 28, 1983)

§ 1. Introduction.

We shall discuss A. Morimoto's problem ([10]) concerned with the tolerance stability conjecture of E. C. Zeeman mentioned in F. Takens ([15]).

Let φ be a (self-) homeomorphism of a compact metric space X with a metric d. A sequence of points $\{x_i\}_{i\in Z}$ is called a δ-pseudo-orbit of φ if $d(\varphi(x_i), x_{i+1})<\delta$ for $i\in \boldsymbol{Z}$. A sequence $\{x_i\}_{i\in Z}$ is called to be ε-*traced* by $x\in X$ if $d(\varphi^i(x), x_i)<\varepsilon$ holds for $i\in \boldsymbol{Z}$. We say that (X, φ) has the *pseudo-orbit tracing property* (abbrev. P.O.T.P.) if for every $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo-orbit of φ can be ε-traced by some point $x\in X$. We know (see A. Morimoto [11] or N. Aoki [2]) that a toral automorphism has P.O.T.P. iff it is hyperbolic.

The set $\mathcal{C}(X)$ of all closed non-empty subsets of X will be a compact metric space by the Hausdorff metric $\bar{d}$ defined by

$$\bar{d}(A, B)=\max\{\max_{b\in B}\min_{a\in A} d(a, b), \max_{a\in A}\min_{b\in B} d(a, b)\}$$

for $A, B\in \mathcal{C}(X)$ (cf. C. Kuratowski [8]). We denote by $\mathrm{Orb}^{\delta}((X, \varphi))$ the set of all δ-pseudo-orbit of φ and by $\widetilde{\mathrm{Orb}}^{\delta}((X, \varphi))$ the set of all $A\in \mathcal{C}(X)$, for which there is $\{x_i\}\in \mathrm{Orb}^{\delta}((X, \varphi))$ such that $A=\mathrm{cl}\{x_i : i\in \boldsymbol{Z}\}$, cl denoting the closure. Let $E(\varphi)$ denote the set of all $A\in \mathcal{C}(X)$ such that for every $\varepsilon>0$ there is $A_\varepsilon\in \widetilde{\mathrm{Orb}}^{\varepsilon}((X, \varphi))$ with $\bar{d}(A, A_\varepsilon)<\varepsilon$. Obviously $E(\varphi)$ is closed in $\mathcal{C}(X)$. On the other hand, we define $O(\varphi)=\mathrm{cl}\{O_\varphi(x) : x\in X\}$ where $O_\varphi(x)=\mathrm{cl}\{\varphi^i(x) : i\in \boldsymbol{Z}\}$. It is clear that $O(\varphi)\subset E(\varphi)$. We call φ to have *OE-property* if $E(\varphi)=O(\varphi)$. It is easy to check that φ has *OE-property* whenever φ has P.O.T.P.

The question whether every toral automorphism with OE-property could be hyperbolic was raised by A. Morimoto ([10]). For this question we can give an answer as follows.

THEOREM. *Let X be a compact metric group and σ be an automorphism of X. If σ has OE-property, then σ has P.O.T.P.*

An easy consequence is the following

COROLLARY. *Every toral automorphism with OE-property is hyperbolic.*

For 2 and 3 dimensional toral automorphisms, the corollary was proved in