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Volumes of tubes about Kähler submanifolds expressed in terms of Chern classes

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1. Introduction.

Let P be a topologically embedded Kähler submanifold of compact closure in a complete Kähler manifold M. Denote by $V_P^M(r)$ the volume of a tube of radius r about P. I shall give inequalities for $V_P^M(r)$ in terms of the Chern classes of P and M that depend on the sectional curvature of M. These inequalities are generalizations of Weyl's formula [**WY**] for the volumes of tubes about submanifolds of Euclidean space.

Let F be the Kähler form of P and denote by $\gamma_c(R^P-R^M)$ the c^{th} Chern form of R^P-R^M, where R^P and R^M are the curvature operators of P and M. Also let K^M denote the sectional curvature of M, and let $n=\dim_{\mathbf{C}}M$, $q=\dim_{\mathbf{C}}P$.

THEOREM 1.1. *Suppose $r>0$ is not larger than the distance from P to its nearest focal point.*

(i) *If $K^M \geqq 0$ then*

$$V_P^M(r) \leqq \sum_{c=0}^{q} \frac{(\gamma_c(R^P-R^M)\wedge F^{q-c})[P]}{(n-q+c)!\,(q-c)!}(\pi r^2)^{n-q+c} \leqq \frac{(\pi r^2)^{n-q}}{(n-q)!}\operatorname{vol}(P). \tag{1.1}$$

(ii) *If $K^M \leqq 0$ then*

$$V_P^M(r) \geqq \sum_{c=0}^{q} \frac{(\gamma_c(R^P-R^M)\wedge F^{q-c})[P]}{(n-q+c)!\,(q-c)!}(\pi r^2)^{n-q+c}. \tag{1.2}$$

(iii) *If M has nonnegative holomorphic bisectional curvature, then*

$$V_P^M(r) \leqq \frac{(\pi r^2)^{n-q}}{(n-q)!}\operatorname{vol}(P). \tag{1.3}$$

COROLLARY 1.2. *If $P\subset \mathbf{C}^n$ is a Kähler submanifold and $r>0$ is not greater than the distance from P to its nearest focal point, then*

$$V_P^{\mathbf{C}^n}(r) = \sum_{c=0}^{q} \frac{(\gamma_c(R^P)\wedge F^{q-c})[P]}{(n-q+c)!\,(q-c)!}(\pi r^2)^{n-q+c}. \tag{1.4}$$