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How many primes decompose completely in an infinite unramified Galois extension of a global field?

Dedicated to Professor I. R. Šafarevič on his 60-th birthday

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1. Let M/k be an infinite unramified Galois extension of a global field k. By investigating an analogue of $d\log\zeta(s)$ (ζ: the zeta function) for the infinite extension field M, and its analytic continuation especially towards $s=1/2$, we obtain an upper bound for some "weighted cardinality" of the set of primes of k that decompose almost completely in M. In the function field case, our upper bound is attained by those M/k which correspond to torsion-free co-compact irreducible discrete subgroups Γ of $PSL_2(\boldsymbol{R})\times PGL_2(F_\mathfrak{p})$ ($F_\mathfrak{p}$: a $\mathfrak{p}$-adic field) (called Γ-classfields in [**4**], and obtained by the reduction mod $\mathfrak{p}$ of towers of Shimura curves cf. also [**3**][**5**]). In a sense, our inequality may be viewed as playing the role of "the second norm index inequality for (non-abelian) classfield theory of Γ-type". In the number field case, we must assume the generalized Riemann hypothesis to obtain an equally good bound. But this conditional result will also be presented, with the hope that its comparison with the situation in the function field case might be suggestive for further study of infinite unramified extensions.

We state our main results in § 2, and give their proofs in §§ 3～12. The crucial part of the proof lies in the study of the limit of

$$[K:k]^{-1}d\log\zeta_K(s)\qquad \begin{matrix} k\subset K\subset M \\ [K:k]<\infty \end{matrix}$$

as $K\to M$, especially in the domain $1/2<\mathrm{Re}(s)<1$ where the Dirichlet series expression for $d\log\zeta_K(s)$ is no longer valid. This is done by careful examination of the property of their inverse Mellin transforms. In the number field case, the "effective analysis" for $d\log\zeta_K(s)$ initiated by Stark [**15**] and continued by Odlyzko, Lagarias, Serre, Poitou, $\cdots$ is crucial. In § 13, we indicate what modifications are necessary if we allow some tame ramifications in M/k. In § 14, we give some examples of Golod-Šafarevič type. The final section § 15 is for various