J. Math. Soc. Japan
Vol. 35, No. 4, 1983

Boolean valued interpretation of Hilbert space theory

By Masanao OZAWA

(Received July 19, 1982)

1. Introduction.

In 1966, D. Scott and R. Solovay reformulated the theory of P. J. Cohen's forcing in terms of Boolean valued models and they also introduced Boolean valued analysis as an application of Boolean valued model theory to analysis. Recently, G. Takeuti developed the Boolean valued analysis extensively in connection with operator theory, harmonic analysis and operator algebras [**6**], [**7**], [**8**], [**9**], [**10**].

In this paper, we study Boolean valued analysis of Hilbert space theory. Let $\mathcal{M}$ be a commutative W^*-algebra and $\mathcal{B}$ the complete Boolean algebra of projections in $\mathcal{M}$. We construct an embedding $H \rightarrow \tilde{H}$ of any non-degenerate normal $*$-representation of $\mathcal{M}$ on a Hilbert space H, which we call a normal $\mathcal{M}$-module H in this paper, in Scott-Solovay's Boolean valued model $V^{(\mathcal{B})}$ of set theory as a Hilbert space $\tilde{H}$ in $V^{(\mathcal{B})}$ and study functorial properties of this embedding. We prove that this embedding is an equivalence between the category of normal $\mathcal{M}$-modules and the category of Hilbert spaces in $V^{(\mathcal{B})}$ and that the multiplicity function of a normal $\mathcal{M}$-module H coincides with the dimension of $\tilde{H}$ in $V^{(\mathcal{B})}$, which is a cardinal in $V^{(\mathcal{B})}$. Thus the Hahn-Hellinger spectral multiplicity theory can be reduced to the Boolean valued interpretation of the simple statement that two Hilbert spaces are isomorphic if and only if they have the same dimension. These results also shed some lights on Takeuti's transfer theorem of von Neumann algebras to factors in $V^{(\mathcal{B})}$ [**10**], where he constructed Hilbert spaces in $V^{(\mathcal{B})}$ by enlarging original Hilbert spaces. In particular, our results improve his machinery in the point that we need not enlarge original Hilbert spaces in order to obtain Hilbert spaces in $V^{(\mathcal{B})}$; only we have to do is to change the truth value of the equality between vectors.

In Section 2, we give necessary preliminaries. In Section 3, it is shown that L^p-spaces on the spectrum of $\mathcal{M}$ is the complex numbers in $V^{(\mathcal{B})}$. Later development depends much on this fact. In Section 4, we construct the embedding $H \rightarrow \tilde{H}$. In Section 5, we prove its functorial properties. In Section 6, the connection with multiplicity theory is established.

The author wishes to express his gratitude to Professor G. Takeuti for the use of his results appearing in his preprint.