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On the unit groups of Burnside rings of finite groups

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Introduction.

Let G be a finite group and $\Theta(G)$ the set of G-isomorphism classes of all finite (left) G-sets. Then $\Theta(G)$ is a semi-ring with addition and multiplication induced by disjoint union and cartesian product, respectively. The Burnside ring $A(G)$ of G is defined to be the Grothendieck ring of $\Theta(G)$. Let $A(G)^*$ be the unit group of the Burnside ring $A(G)$.

In this note we shall study $A(G)^*$ and the homomorphism $u: RO(G)\rightarrow A(G)^*$, where $RO(G)$ is the real representation ring of G and u is the homomorphism defined by T. tom. Dieck (see 1.2). By the famous theorem of Feit-Thompson (G is solvable if $|G|$ is odd) and by a result of A. Dress (idempotents of $A(G)$ are determined by perfect subgroups of G, cf. [1] Proposition 1.4.1), we know that

$$|A(G)^*|=2 \qquad \text{if } |G| \text{ is odd}$$

(cf. [1] Proposition 1.5.1). Therefore, it remains to study $A(G)^*$ and the homomorphism $u: RO(G)\rightarrow A(G)^*$ for groups G of even order.

In Section 1, we describe the well known results for $A(G)^*$ and the homomorphism $u: RO(G)\rightarrow A(G)^*$.

Section 2 is the main part of this note, and we obtain the following Theorem A and Theorem B.

THEOREM A (cf. Theorem 2.2, Corollary 2.4 and Lemma 2.5). *$u: RO(G)\rightarrow A(G)^*$ is surjective if and only if $u: RO(G')\rightarrow A(G')^*$ is surjective for every homomorphic image G' of G such that $|C(G')|\leqq 2$, where $C(G')$ is the center of G'.*

THEOREM B (cf. Theorem 2.9 and Theorem 2.11). *Let $1\rightarrow H\rightarrow G\rightarrow K\rightarrow 1$ be a group extension. Then we have*

(i) *K acts on $A(H)^*$* (cf. 2.6) *and $\mathrm{Res}_H^{G*}(A(G)^*)\subset(A(H)^*)^K$, where Res_H^{G*} is the natural restriction homomorphism from $A(G)^*$ to $A(H)^*$,*

(ii) *if $|K|$ is odd and $u: RO(H)\rightarrow A(H)^*$ is surjective, then $u: RO(G)\rightarrow A(G)^*$ is surjective and $\mathrm{Res}_H^{G*}: A(G)^*\rightarrow(A(H)^*)^K$ is an isomorphism,*

(iii) *if the group extension is split and $|K|$ is odd, then $\mathrm{Res}_H^{G*}: A(G)^*\rightarrow(A(H)^*)^K$ is an isomorphism.*