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Some statement which implies the existence of Ramsey ultrafilters on ω

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1. Introduction and results.

Throughout this paper, we work in Zelmero-Fraenkel set theory with choice (ZFC). Let $\mathfrak{F}$ be a filter on A and let f be a function from A to B. $f(\mathfrak{F})$ denotes the filter $\{y\subset B; f^{-1}y\in\mathfrak{F}\}$. $\mathfrak{F}$ is said to be *free* if $\emptyset\notin\mathfrak{F}$ and $\bigcap\mathfrak{F}=\emptyset$. $\mathfrak{F}$ is said to be *ample* if there exists an infinite subset A_0 of A such that, for any $x\in\mathfrak{F}$, A_0-x is finite. $\mathfrak{F}$ is said to be *weakly ample* if, for any free ultrafilter (uf) $\mathfrak{U}$ on ω, there exists a function g from ω to A such that $g(\mathfrak{U})\supset\mathfrak{F}$. It is trivial that any free, ample filter is weakly ample. For any infinite cardinal κ, we denote by AN(κ) the statement: "any free, weakly ample filter on κ is ample". It is easy to see that, whenever $\kappa\leqq\lambda$, AN(λ) implies AN(κ). Puritz proved the following Theorem 1.

THEOREM 1 (Puritz [**5**]).

(a) *The continuum hypothesis* (CH) *implies* AN(c), *where c denotes* 2^{ω}.

(b) AN(ω) *implies that there are P-points on* ω.

(c) AN(2^c) *does not hold.*

He asked whether the existence of P-points implies AN(ω). This question is answered negatively by Theorem 5 which appears below. By Theorem 1 (a), (c), under the assumption CH$+2^{\omega_1}=\omega_2$, AN(κ) holds if and only if $\kappa=\omega$ or $\kappa=\omega_1$. Let P be the statement: "any free, κ-generated filter on ω is ample, for all $\kappa<c$". Then, the proof of Theorem 1 (a) (in [**5**; p. 222]) yields a proof of that P implies AN(c). Since Martin's Axiom (MA) implies P (cf. [**4**; Theorem 5]), it holds that MA implies AN(c). By this, Theorem 1 (b) and a result of Shelah that the existence of P-points is unprovable (in ZFC), the negation of CH implies neither AN(c) nor $\neg$AN(c).

We shall consider what cardinals κ satisfy AN(κ) in the cases where CH $+2^{\omega_1}=\omega_2$ fails. Our results are the following Theorems which are proved in Sections 3~6.

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