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Boundaries of the Teichmüller spaces of finitely generated Fuchsian groups of the second kind

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The Teichmüller space $T(\Gamma)$ of a Fuchsian group Γ can be embedded, as a bounded domain, into the Banach space $B(\Gamma)$ of bounded quadratic differentials for Γ. Hence the boundary $\partial T(\Gamma)$ of $T(\Gamma)$ can be defined naturally. The boundary $\partial T(\Gamma)$ was investigated by Bers [**Ber**], Maskit [$\mathbf{M}_3$], Abikoff [**A**] and others. However, most of them were done under the restriction that Γ is a finitely generated Fuchsian group of the first kind, that is, $T(\Gamma)$ is finitely dimensional. In this paper we investigate the boundaries of Teichmüller spaces of finitely generated Fuchsian group of the second kind. In this case the dimension of $T(\Gamma)$ is infinite. For each $\phi\in T(\Gamma)\cup\partial T(\Gamma)$ the meromorphic homeomorphism W_ϕ of the lower half plane L can be defined, and induces an isomorphism $\chi_\phi : \Gamma\to W_\phi\Gamma W_\phi^{-1}$. A point $\phi\in\partial T(\Gamma)$ is called a cusp if a hyperbolic element is mapped to a parabolic one under χ_ϕ. The existence of cusps was proved by Bers [**Ber**] if Γ is a finitely generated Fuchsian group of the first kind. In section 2 we prove the existence of cusps even if Γ is a finitely generated Fuchsian group of the second kind. As usual we can prove the above statement by obtaining cusps as limits of sequences in $T(\Gamma)$ obtained by squeezing deformations, the definition of which and necessary preliminaries are exhibited in § 1. In § 3 we prove that if $\phi\in\partial T(\Gamma)$ is not a cusp, then $\chi_\phi(\Gamma)$ is a quasi-Fuchsian group, which never exists on the boundaries of the Teichmüller spaces of finitely generated Fuchsian groups of the first kind. By using the method in the proof of this theorem, we also give an alternative proof of the existence of cusps and of the estimate of outradii of the Teichmüller spaces of parabolic or finite cyclic groups, which are due to Sekigawa [$\mathbf{Se}_1$], [$\mathbf{Se}_2$]. Finally, in § 4 we prove the existence of geometrically infinite cusps.

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