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# Equational theories and universal theories of fields

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## 1. Introduction.

In this paper, we will investigate equational theories and universal theories of fields. To begin with, we will explain how to deal with problems connected with the multiplicative inverse in field theory. The multiplicative inverse $x^{-1}$ on a given field is defined for all $x$ but zero element 0, so the function $^{-1}$ will be regarded as a partial function. On the other hand, in the customary treatment in logic, every function symbol is interpreted as a total function on a given structure. Thus, the language $\mathcal{L}$ for ring theory which consists of $\{+, -, \cdot, 0, 1\}$ will be usually employed when we formulate the theory of fields. In this language, the existence of the inverse will be represented as

$$(1) \qquad \forall x \exists z(\neg(x=0) \rightarrow xz=1).$$

But, if we will restrict our attention only to universal theories of fields, then we will have to deal with not only the class of fields but also some broader class of algebraic structures, since (1) can not be expressed by a universal formula, i. e., a formula of the form $\forall x_1 \cdots \forall x_n\, \varphi(x_1, \cdots, x_n)$ for some open formula $\varphi(x_1, \cdots, x_n)$. In the above case, we can take the class of integral domains for this, because the axioms of integral domains, which we denote by $\Theta$ in the following, can be expressed by universal formulas and moreover it can be shown that the set of universal formulas of $\mathcal{L}$ valid in all fields (of characteristic $p$) is equal to the set of universal formulas valid in all integral domains (of characteristic $p$).

On the other hand, it will be possible to treat equational theories of fields, if we determine the value of $0^{-1}$ in any way. In this way, Komori introduced in [**9**] the notion of *pseudo-fields* and proved that for any equation if it holds in every skew field then it holds also in every pseudo-field and vice versa. In particular, he introduced the notion of *desirable fields*, the skew fields in which $0^{-1}=0$ holds. Following his idea, we will introduce an axiom system $\Sigma$ for the equational theory of commutative regular rings in the language $\mathcal{L}'=\{+, -, \cdot, 0, 1, ^{-1}\}$. Thus, we will study equational theories of (commutative)

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