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On G-functors (II): Hecke operators and G-functors

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1. Introduction.

There are some facts which suggest the relation between cohomological G-functors and Hecke rings. In cohomology theory of finite groups, Cline, Parshall and Scott showed that Hecke algebras act on cohomology groups of finite groups and described the stability theorem of Cartan-Eilenberg (a generalization of the focal subgroup theorem for finite groups) by the use of the language of fixed points (=common eigenvectors) for an action of the Hecke algebra ([**1**, Section 6]). They pointed out also an example that there is no corresponding Hecke algebra action for algebraic K-theory. It is stated also in [**3**], [**4**, Section 8.3], etc. that Hecke algebras act on cohomology groups of groups.

The purpose of this paper is to study the relation between cohomological G-functors and representation of Hecke rings. In Section 2, we define G-functors. In Section 3, we introduce the concept of Hecke category which is the category of permutation modules. In application, Hecke rings appear frequently as components of additive functors from the Hecke category. In Section 4, it is proved that the concepts of cohomological G-functors and representations of the Hecke category (that is, additive functors from there) are equivalent. The main theorem of this paper is the following:

THEOREM 4.3. *Let $\mathcal{M}_k(G)^c$ be the category of cohomological G-functors over k and let $\mathcal{H}_{kG}$ be the category of permutation modules $k[G/H]$, $H\leq G$. Then*

$$\mathcal{M}_k(G)^c \cong \mathrm{Add}_k(\mathcal{H}_{kG}, \mathcal{M}_k)\,.$$

In general, if $\boldsymbol{a}$ be a cohomological G-functor, then the action of the Hecke ring $k[H\backslash G/H]$ on $\boldsymbol{a}(H)$ is given by

$$\alpha\cdot(HxH):=\alpha^x{}_{Hx\cap H}{}^H\,, \qquad \alpha\in\boldsymbol{a}(H)\,,\quad x\in G\,.$$

NOTATION. G denotes a finite group except for Section 5. "$H\leq G$" means that H is a subgroup of G. For a subset or an element X of G and an element g of G, we set $X^g:=g^{-1}Xg:=\{g^{-1}xg\,|\,x\in X\}$. The index of a subgroup H of G is denoted by $|G:H|$. The cardinality of a set X is denoted by $|X|$. The notation $H\backslash G/K$ means the set of double cosets HgK and sometimes a complete