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Ricci curvature, geodesics and some geometric properties of Riemannian manifolds with boundary

By Atsushi KASUE

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Introduction.

Let M be a connected, complete Riemannian manifold with (possibly empty) boundary ∂M. Cheeger and Gromoll proved in [**4**] that if ∂M is empty and the Ricci curvature of M is nonnegative, then the Busemann function with respect to any ray is superharmonic on M. From this result, they showed that M as above is the isometric product $N\times \boldsymbol{R}^k$ $(k\geqq 0)$, where N contains no lines and $\boldsymbol{R}^k$ has its standard flat metric. They also proved in [**5**] that if M is a convex subset with boundary ∂M in a positively curved manifold, then the distance function to ∂M is concave on M. Later, making use of this result, Burago and Zalgaller obtained in [**3**] a theorem on such a manifold M saying that

(1) the number of components of ∂M is not greater than 2,

(2) if there are two components Γ_1 and Γ_2 of ∂M, then M is isometric to the direct product $[0, a]\times\Gamma_1$,

(3) if ∂M is connected and compact, but M is noncompact, then M is isometric to the direct product $[0, \infty)\times\partial M$.

Recently we have obtained in [**9**] a sharp and general Laplacian comparison theorem, which tells us the behavior of the Laplacian of a distance function or a Busemann function on M in terms of the Ricci curvature of M. In this paper, using our comparison theorem, we shall study Riemannian manifolds with boundary and obtain, roughly speaking, a generalization of the above result by Burago and Zalgaller from the viewpoint of Ricci curvature.

We shall now describe our main theorems. Let M be a connected, complete Riemannian manifold of dimension m with smooth boundary ∂M. We call M *complete* if it is complete as a metric space with the distance induced by the Riemannian metric of M. Let R and Λ be two real numbers. We say M is *of class* (R, Λ) if the Ricci curvature of $M\geqq(m-1)R$ and (the trace of $S_\xi)\leqq(m-1)\Lambda$ for any unit inner normal vector field ξ of ∂M, where S_ξ is the second fundamental form of ∂M with respect to ξ (i.e., $\langle S_\xi X, Y\rangle=\langle\nabla_X\xi, Y\rangle$). We write $i(M)$ for the inradius of M (i.e., $i(M)=\sup\{\mathrm{dis}_M(x, \partial M): x\in M\}\leqq+\infty$). Let f