Ricci curvature, geodesics and some geometric properties of Riemannian manifolds with boundary

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Introduction.

Let $M$ be a connected, complete Riemannian manifold with (possibly empty) boundary $\partial M$. Cheeger and Gromoll proved in [4] that if $\partial M$ is empty and the Ricci curvature of $M$ is nonnegative, then the Busemann function with respect to any ray is superharmonic on $M$. From this result, they showed that $M$ as above is the isometric product $N \times R^k (k \geq 0)$, where $N$ contains no lines and $R^k$ has its standard flat metric. They also proved in [5] that if $M$ is a convex subset with boundary $\partial M$ in a positively curved manifold, then the distance function to $\partial M$ is concave on $M$. Later, making use of this result, Burago and Zalgaller obtained in [3] a theorem on such a manifold $M$ saying that

1. the number of components of $\partial M$ is not greater than 2,
2. if there are two components $\Gamma_1$ and $\Gamma_2$ of $\partial M$, then $M$ is isometric to the direct product $[0, a] \times \Gamma_1$,
3. if $\partial M$ is connected and compact, but $M$ is noncompact, then $M$ is isometric to the direct product $[0, \infty) \times \partial M$.

Recently we have obtained in [9] a sharp and general Laplacian comparison theorem, which tells us the behavior of the Laplacian of a distance function or a Busemann function on $M$ in terms of the Ricci curvature of $M$. In this paper, using our comparison theorem, we shall study Riemannian manifolds with boundary and obtain, roughly speaking, a generalization of the above result by Burago and Zalgaller from the viewpoint of Ricci curvature.

We shall now describe our main theorems. Let $M$ be a connected, complete Riemannian manifold of dimension $m$ with smooth boundary $\partial M$. We call $M$ complete if it is complete as a metric space with the distance induced by the Riemannian metric of $M$. Let $R$ and $A$ be two real numbers. We say $M$ is of class $(R, A)$ if the Ricci curvature of $M \equiv (m-1)R$ and (the trace of $S_\xi \leq (m-1)A$ for any unit inner normal vector field $\xi$ of $\partial M$, where $S_\xi$ is the second fundamental form of $\partial M$ with respect to $\xi$ (i.e., $\langle S_\xi X, Y \rangle = \langle \nabla_X \xi, Y \rangle$). We write $i(M)$ for the inradius of $M$ (i.e., $i(M) = \sup \{ \text{dis}_M(x, \partial M) : x \in M \} \leq +\infty$). Let $f$