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Von Neumann algebras and Boolean valued analysis

By Gaisi TAKEUTI

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The aim of this paper is to apply Boolean valued analysis developed in [**5**], [**6**], [**7**], [**8**] to von Neumann algebras. Let $\mathcal{M}$ be a von Neumann algebra, $\mathcal{Z}$ its center, and $\mathcal{B}$ the complete Boolean algebra of all projections in $\mathcal{Z}$. Then $\mathcal{B}$-valued analysis gives a machinery to transfer theorems on factors into theorems on von Neumann algebras whose centers are $\mathcal{Z}$. Our machinery is a kind of a dictionary which gives a translation of the notions on factors into the notions on von Neumann algebras with the center $\mathcal{Z}$. E. g., the translations of type I, type II_1, type II_∞ and type III are exactly type I, type II_1, type II_∞ and type III themselves. However, the translations of trace and weight become generalized center valued trace and generalized center valued weight respectively. This machinery immediately reduces many properties of center valued traces or weights to properties of traces or weights on factors. In § 5, we state direct translations of Murray and von Neumann's theorem on factors of type II_∞ and Takesaki's theorem on factors of type III in order to show how our machinery works.

Our theory is closely related to von Neumann's reduction theory. We prove that every von Neumann algebra is a factor in a Boolean valued sense while the reduction theory proves that every von Neumann algebra with a countability condition is a direct integral of factors. Thus a systematic interpretation of Boolean valued notion provides us with a machinery of automatic translations of notions on factors into notions on von Neumann algebras with the center $\mathcal{Z}$, while the reduction theory reduces many problems on von Neumann algebras to problems on factors.

It is not difficult to eliminate $\mathcal{B}$-valued analysis in our machinery. In order to do so, we have to introduce the following $\bar{\mathcal{Z}}$-valued Hilbert space, where $\bar{\mathcal{Z}}$ is the unbounded extension of $\mathcal{Z}$. If we express $\mathcal{Z}$ as $L^\infty(\Omega, \mu)$, then $\bar{\mathcal{Z}}$ is the set of all measurable functions. $\hat{\mathcal{H}}$ is called a $\bar{\mathcal{Z}}$-valued Hilbert space if $\hat{\mathcal{H}}$ is a $\bar{\mathcal{Z}}$-module with the inner product satisfying the following properties:

1) $\zeta, \eta \in \hat{\mathcal{H}} \Rightarrow (\zeta | \eta) \in \bar{\mathcal{Z}}$
2) Let $\zeta, \eta, \xi \in \hat{\mathcal{H}}$ and $f, g \in \bar{\mathcal{Z}}$. Then the following hold.

2.1) $(f\zeta | \eta) = f \cdot (\zeta | \eta)$ a. e.