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A characterization of Azumaya coalgebras over a commutative ring

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§1. Introduction.

Throughout this paper R is a commutative ring with 1, and (C, Δ, ε) is a coalgebra over R, where Δ is the comultiplication of C and ε is the counit of C. As usual we denote $\Delta(c)=\sum c_{(1)}\otimes c_{(2)}$ for each $c\in C$. Furthermore we will set $C^*=\mathrm{Hom}_R(C, R)$, and for each $c^*\in C^*$ and $c\in C$, we denote by $\langle c^*, c\rangle$ the element of R to which c is mapped by c^* in stead of $c^*(c)$. As is well known C^* is an R-algebra whose multiplication is defined by $\langle c^*\cdot d^*, c\rangle=\sum\langle c^*, c_{(1)}\rangle\langle d^*, c_{(2)}\rangle$ (namely, $(c^*\cdot d^*)(c)=\sum c^*(c_{(1)})d^*(c_{(2)})$ by the ordinary description of homomorphisms) for any c^*, $d^*\in C^*$ and $c\in C$. On the other hand, C is a two-sided C^*-module by $c^*\cdot c=\sum c_{(1)}\langle c^*, c_{(2)}\rangle$ and $c\cdot c^*=\sum\langle c^*, c_{(1)}\rangle c_{(2)}$ for any $c^*\in C^*$ and $c\in C$. Then it is easily seen that the C^*-C^*-module structure of $\mathrm{Hom}_R(C, R)$ induced from the C^*-C^*-module structure of C is the same as that induced from the ring structure of $\mathrm{Hom}_R(C, R)=C^*$. In what follows throughout, all $\otimes$ will be $\otimes_R$ and Hom will mean Hom_R.

In this paper we will show that in the case where C is R-finitely generated projective and faithful, C^* is an R-Azumaya algebra if and only if there exist C^*-C^*-isomorphisms Ψ of $C\otimes C$ to $C\otimes_{C^*}C\otimes C$ and μ of $C^*\otimes I$ to C, where $I=\{c\in C\mid \sum c_{(1)}\otimes c_{(2)}=\sum c_{(2)}\otimes c_{(1)}\}$, such that $\Psi(c\otimes d)=\sum c\otimes d_{(1)}\otimes d_{(2)}$ and $\mu(c^*\otimes a)=c^*\cdot a$ $(=a\cdot c^*)$ for c, $d\in C$, $c^*\in C^*$ and $a\in I$

§2. Let A, B and S be (not necessarily commutative) rings with identities. We denote as usual ${}_AM_B$ (resp. $M_{A\text{-}B}$) in the case where M is a left A-module as well as a right B-module (resp. a right A-module as well as a right B-module) such that $(am)b=a(mb)$ (resp. $(ma)b=(mb)a$) for all $m\in M$, $a\in A$ and $b\in B$. For any ${}_AP_A$ and ${}_AM_B$, ${}_AN_B$, we will set, respectively,

$$P^A=\{x\in P\mid ax=xa \text{ for all } a\in A\},$$

$$\mathrm{Hom}({}_AM_B, {}_AN_B)=\{A\text{-}B\text{-homomorphism of } M \text{ to } N\}.$$

Then it is clear that $\mathrm{Hom}({}_AM_B, {}_AN_B)=[\mathrm{Hom}(M_B, N_B)]^A=[\mathrm{Hom}({}_AM, {}_AN)]^B$. The