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On isometry groups of a manifold without focal points

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Introduction.

This note treats a complete simply connected Riemannian manifold without focal points. We know that the class of all such manifolds is wider than that of all complete simply connected Riemannian manifolds with nonpositive sectional curvature (cf. [**10**], [**6**] and [**7**]). The former is defined by the property: No maximal geodesic has focal points along any perpendicular geodesic, or equivalently, there is a unique perpendicular geodesic from a point to a geodesic, see Proposition 2 in [**3**]. It has been observed that several results on manifolds with nonpositive sectional curvature hold for manifolds without focal points. In this note we shall, furthermore, investigate the problems of this nature.

In §1 we summarize notion and known facts which would be used later. The main object of §2 is to study about the behavior of a hyperbolic isometry and its fixed points. In manifolds with nonpositive curvature the law of cosine is powerful especially for the limit process of divergent sequences. We shall give here a crucial lemma which plays the same role as the special use of the law of cosine. The main result in §2 is

THEOREM 1. *Let M be a complete simply connected Riemannian manifold without focal points and Γ a freely acting, properly discontinuous group of isometries of M. Let D be a Dirichlet region for Γ. If $z\in M(\infty)$ is a point containing the axis of a hyperbolic element in Γ, then $\Gamma(z)\cap\bar{D}=\emptyset$.*

§3 studies the limit sets of freely acting, properly discontinuous groups Γ of isometries of M, as above, with volume $(M/\Gamma)<\infty$. §4 is devoted to extend E. Cartan's fixed point theorem proved in the case of nonpositive curvature, which leads to the conjugacy of maximal compact subgroups of a (semisimple) Lie group, to the case without focal points.

It is not yet known if the displacement function of an isometry of M, as above, is convex. If we assume the property, together with our results one could see actions of isometries of M more clearly.