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On the units of an algebraic number field

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In this paper, we extend the transcendental method of Ax [1], to apply the result of Brumer [2] to show Leopoldt's conjecture for certain non-abelian extensions of imaginary quadratic number fields (Theorem 4 in §6).

§1. Preliminaries.

Let F be a finite algebraic extension of rational number field $\boldsymbol{Q}$, and O_F the maximal order of F. For a prime divisor $\mathfrak{p}$ of F, let $F_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of F, and $O_{\mathfrak{p}}$ the closure of O_F in $F_{\mathfrak{p}}$.

Let p be a prime number, and denote the p-adic completion of $\boldsymbol{Q}$ by $\boldsymbol{Q}_p$. The closure of the ring of integers $\boldsymbol{Z}$ in $\boldsymbol{Q}_p$ is denoted by $\boldsymbol{Z}_p$. Then $F\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$ is naturally isomorphic to the direct sum $\bigoplus_{\mathfrak{p}|p} F_{\mathfrak{p}}$.

We denote the multiplicative groups of the invertible elements of F, $F_{\mathfrak{p}}$, $O_{\mathfrak{p}}$, etc. by $F^{\times}$, $F_{\mathfrak{p}}^{\times}$, $O_{\mathfrak{p}}^{\times}$, etc. Especially, $(\bigoplus_{\mathfrak{p}|p} F_{\mathfrak{p}})^{\times}$ is the direct product $\prod_{\mathfrak{p}|p} F_{\mathfrak{p}}^{\times}$. Let $W_{\mathfrak{p}}$ be the group of $(N_{F/\boldsymbol{Q}}(\mathfrak{p})-1)$-th roots of 1 in $F_{\mathfrak{p}}$. Then $O_{\mathfrak{p}}^{\times}=W_{\mathfrak{p}}\cdot(1+\mathfrak{p}\cdot O_{\mathfrak{p}})$. Put $U_0=\prod_{\mathfrak{p}|p} O_{\mathfrak{p}}^{\times}$ and $U_1=\prod_{\mathfrak{p}|p}(1+\mathfrak{p}\cdot O_{\mathfrak{p}})$. The action of $\boldsymbol{Z}$ on the compact abelian group U_1 as powers induces the action of $\boldsymbol{Z}_p$ on U_1 naturally. As a $\boldsymbol{Z}_p$-module in this way, the essential rank of U_1 over $\boldsymbol{Z}_p$ is equal to $[F:\boldsymbol{Q}]$, the degree of F over $\boldsymbol{Q}$. In other words, the dimension of the vector space $U^{(p)}=U_1\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=U_1\otimes_{\boldsymbol{Z}_p}\boldsymbol{Q}_p$ over $\boldsymbol{Q}_p$ is $[F:\boldsymbol{Q}]$. Note that $U_0\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=U_1\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=U^{(p)}$.

Let V_0 be a finitely generated subgroup of $F^{\times}\cap U_0$. Here $F^{\times}$ is considered to be diagonally imbedded in $\prod_{\mathfrak{p}|p} F_{\mathfrak{p}}^{\times}$. Put $V=V_0\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, and $V^{(p)}=V\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$. Then the inclusion map $V_0\hookrightarrow U_0$ induces a $\boldsymbol{Q}_p$-linear map $\Phi_p: V^{(p)}\to U^{(p)}$. We are interested in the dimension over $\boldsymbol{Q}_p$ of the subspace $\Phi_p(V^{(p)})$ of $U^{(p)}$. (Leopoldt's conjecture is equivalent to the injectivity of Φ_p for $V_0=O_F^{\times}=$ the group of the units of F.) Note that

$$\dim_{\boldsymbol{Q}_p} V^{(p)}=\dim_{\boldsymbol{Q}} V=\text{ess. rank}_{\boldsymbol{Z}} V_0\,,$$

and that $\Phi_p|_V: V\to U^{(p)}$ is injective.

We use additive notation for the vector spaces V, $V^{(p)}$, and $U^{(p)}$.