J. Math. Soc. Japan
Vol. 34, No. 3, 1982

Hadamard's variation of the Green kernels of heat equations and their traces I

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(Received Feb. 7, 1980)
(Revised Dec. 3, 1980)

§1. **Introduction.** Let Ω be a bounded domain in R^n with smooth boundary γ. Let $\rho(x)$ be a C^∞ real valued function on γ and ν_x be the exterior unit normal vector at $x\in\gamma$. For any sufficiently small $\varepsilon\geqq 0$, let Ω_ε be the bounded domain whose boundary γ_ε is defined by

$$\gamma_\varepsilon=\{x+\varepsilon\rho(x)\nu_x\,;\,x\in\gamma\}.$$

Let $G_\varepsilon(x, y)$ be the Green's function of the Dirichlet boundary value problem for the Laplacian, that is, $G_\varepsilon(x, y)$ has the following properties:

$$\begin{cases} -\Delta_x G_\varepsilon(x, y)=\delta(x-y) & x,\ y\in\Omega_\varepsilon \\ G_\varepsilon(x, y)=0 & x\in\gamma_\varepsilon,\ y\in\Omega_\varepsilon\,. \end{cases}$$

We abbreviate $G_0(x, y)$ as $G(x, y)$. For any x, $y\in\Omega$ satisfying $x\neq y$, we put

$$\delta G(x, y)=\lim_{\varepsilon\to 0}\varepsilon^{-1}(G_\varepsilon(x, y)-G(x, y))\,.$$

Then the celebrated Hadamard variational formula is the following:

$$\delta G(x, y)=\int_\gamma \frac{\partial G(x, z)}{\partial\nu_z}\frac{\partial G(y, z)}{\partial\nu_z}\rho(z)d\sigma_z\,, \tag{1.1}$$

where $\partial/\partial\nu_z$ denotes the exterior normal derivative with respect to z and $d\sigma_z$ denotes the surface element of γ at z.

In [7], Hadamard proved the formula (1.1) in the case that $\rho(z)$ did not change its sign. And he also proved it when γ was of class C^ω.

Proof of the formula (1.1) for general $\rho(z)\in C^\infty(\gamma)$ can be found, for example, in Garabedian [5], Garabedian-Schiffer [6]. Based on (1.1), many authors derived interesting facts about the Green's function and the results about the theory of functions of one complex variable. See Bergmann-Schiffer [2] and Schiffer-Spencer [12]. Recently, new applications of the formula (1.1) have appeared.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 574061), Ministry of Education.