J. Math. Soc. Japan
Vol. 34, No. 2, 1982

Uniform ascent and descent of bounded operators

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(Received Feb. 4, 1980)
(Revised Oct. 6, 1980)

1. Introduction.

In this paper we study the structure and perturbation theory of certain classes of bounded operators on a Banach space. These classes contain the semi-Fredholm operators, and also most of the generalizations of Fredholm operators which appear in the literature. For a bounded operator T, our study focuses on the sequences of ranges, $\{R(T^n)\}$, and of null-spaces, $\{N(T^n)\}$, and on the analogous sequences for small or compact perturbations of T. We are particularly interested in the spaces

$$R(T^{\infty})=\bigcap_n R(T^n) \qquad \text{and} \qquad N(T^{\infty})=\bigcup_n N(T^n)\,, \tag{1.1}$$

and the analogous spaces for perturbations of T. The results we prove will be similar to some results which have been useful in spectral theory [**21**], [**12**], in the structure theory of Banach algebras [**10**], [**23**], and in the study of automatic continuity [**13**], [**22**].

If T is a bounded linear operator on the Banach space X, then, for each nonnegative integer n, T induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. We will let $k_n(T)$ be the dimension of the null space of the induced map and let

$$k(T)=\sum_0^{\infty} k_n(T)\,. \tag{1.2}$$

The following definition describes the classes of operators we will study.

DEFINITION (1.3). If there is a nonnegative integer d for which $k_n(T)=0$ for $n\geqq d$ (i.e., if the induced maps are isomorphisms for $n\geqq d$), we say that T has *eventual uniform descent*; and, more precisely, that T has *uniform descent for* $n\geqq d$. If $k(T)$ is finite, we say that T has *almost uniform descent*.

We will see, in Lemma (2.3), that $k_n(T)$ is also the dimension of the cokernel of the map induced by T from $N(T^{n+2})/N(T^{n+1})$ to $N(T^{n+1})/N(T^n)$. Thus, some

(1) Research partially supported by National Science Foundation grant MCS 76-0 7000 A01.