J. Math. Soc. Japan
Vol. 34, No. 2, 1982

A condition for holomorphic maps of C^2 into C^2 to be algebraic

By Syoshi TOKUNAGA

(Received Sept. 3, 1980)

1. In this paper we shall give a condition for holomorphic maps of C^2 into C^2 to be algebraic.

DEFINITION. A polynomial $P(X, Y)$ is said to be *of type* (g, n), if the level curve $P_c:=\{(X, Y)\in C^2 \mid P(X, Y)=c\}$ is of genus g and has n boundary points in the two dimensional projective space P^2 for almost every $c\in C$. In particular $P(X, Y)$ is said to be of general type, if $g\geqq 1$ or $n\geqq 3$.

THEOREM. *Let*

$$\Phi: X=f(x, y), \qquad Y=g(x, y)$$

be a holomorphic map of C^2 *into* C^2, *where* $f(x, y)$ *and* $g(x, y)$ *are entire functions. If there exists a polynomial* $P(X, Y)$ *of general type such that the composite function*

$$q(x, y):=P[f(x, y), g(x, y)]$$

is a polynomial, then $f(x, y)$ *and* $g(x, y)$ *are polynomials.*

REMARK. For a polynomial $P(X, Y)$ of type (0, 1) or (0, 2) the theorem is incorrect. In fact, we have the following counterexamples:

1) $P(X, Y)=X$ when $f(x, y)$ is a polynomial and $g(x, y)$ is a transcendental entire function.

2) $P(X, Y)=XY$ when $f(x, y)=e^x$ and $g(x, y)=e^{-x}$.

REMARK. T. Kizuka ([1]) proved the above theorem under the condition that Φ is an automorphism of C^2.

The author would like to thank Professor T. Nishino for suggesting him the problem and for continuous encouragement.

2. Key lemma.

We introduce a lemma which is essential in this paper.

LEMMA (Nishino [2], see also [1]). *Let* (V, π, Γ) *be an analytic family of compact analytic curves of genus g on the disc* Γ: $|z|<1$. *Suppose every fibre on* $z\neq 0$ *is irreducible, non-singular and of genus g. If an unramified finitely many-valued analytic section* η *on the punctured disc* Γ': $0<|z|<1$ *satisfies one*